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A Non-Linear Shell Theory Compared with the Classical Three Dimensional Theory of Elasticity

Chester B. Sensenig

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CONTENTS

	Introduction	1
1.	Pseudo-tensor notation	4
2.	Introduction to the classical theory and shell theory.	9
3.	General comparison of the shell theory and classical theory.	16
4.	Certain dependencies.	18
5.	The parameters used in making estimates, miscellaneous notation and results.	23
6.	An estimate for the L_2 norms of U_1 and its derivatives.	34
7.	L_2 estimates for functions in the set $\dot{M}^{(k)}$.	43
8.	L_2 estimates for functions in a_k and b_k $(k=2,3)$.	52
9.	L_2 estimates for functions in a_4 and b_4.	60
10.	Pointwise estimates for functions in a_k and b_k.	69
11.	Special comparison of the shell theory with the classical theory, some improved estimates, and the importance of the low degree terms in the displacements.	72
	Appendix.	81

ABSTRACT

A non-linear theory for the equilibrium deformation of homogeneous isotropic shells is derived and compared with the classical three dimensional non-linear theory of elasticity.

To obtain the shell theory, let U^1 and U^2 be displacements tangent to the undeformed middle surface and let U^3 be a displacement normal to the middle surface. The displacement fields considered are restricted by requiring that U^1 and U^2 be nth degree polynomials in Θ_3 and U^3 be an (n+1)st degree polynomial in Θ_3 where Θ_3 is the undeformed distance to the middle surface along a normal line. Restricting the displacements in this way, the potential energy of the shell becomes a functional of the 3n+4 coefficients of the displacement polynomials. Requiring that the potential energy be stationary with respect to variations of the 3n+4 coefficients, gives the equilibrium equations and surface traction boundary conditions of the shell theory.

The shell theory is compared with the classical three dimensional theory by examining the errors which result when displacements satisfying the equilibrium equations and surface traction boundary conditions of the shell theory are substituted into those of the classical three dimensional theory. Let E_q, E_e and E_f denote any error resulting from an equilibrium equation, a surface traction boundary condition at the edge, and a surface traction boundary condition at a

face, respectively, of the classical theory. It is shown that E_q, E_e and their derivatives have a stated number of zeros along each line normal to the middle surface, the number of zeros depending on n and the number of differentiations of the error with respect to θ_3. Furthermore, if the shell thickness and deformation are small enough, and if the body forces and surface tractions at the faces and their derivatives are small enough, then, at points not too near the edge, E_f and its derivatives are significantly small if n is large enough, and E_q and its derivatives are significantly small throughout the thickness if n is large enough and there are not too many differentiations of E_q with respect to θ_3. Also, under the previous restrictions, the low degree terms in the displacement polynomials and their derivatives are more significant than the high degree terms at points not too near the edge (at least this is always true if the difference in degrees of the two terms is greater than or equal two).

NOTATION

Page	
1	θ_i (see page 18 also), $\vec{r}(\theta_1,\theta_2)$, $\vec{g}_i$, $\vec{u}$, U^i, n
4	x_i, h, $\vec{x}$
5	g_{ij}, g^{ij}, $\vec{g}^i$, a^i_j, A^i_j
7	b_{ij}, b^i_j, B^i_j, H, K, C, Γ^k_{ij}
9	u_i, e_{ij}, s_i, W, λ, μ
10	t_{ij}, z_{ij}, q_{ij}, $\bar{q}_{i+}$, $\bar{q}_{i-}$, f_i, $\bar{q}_{ie}$
11	n_i, Q^{ij}, $\bar{Q}^i{}_+$, $\bar{Q}^i{}_-$, $\bar{Q}^i{}_e$, N_i, F^i, $\mathcal{E}$
13	$\mathcal{a}$, s
15	$\bar{Q}^i$
18	L_{ij}, M_{ij}
19	S_{ij}, V_i, a_k, b_k
23	f, D, R
24	θ_0, θ, ε
25	d
26	p_{ij}, E^i_j, T^i_j, Z^i_j, $P^i{}_j$, R_{ij}
27	G_i
29	$H_k(A)$, $J_k(A)$
31	U, L, M, E, T, P, Q, R, S, V, F, $\bar{Q}$
32	ζ, ζ_k
34	A_k, B_k

INTRODUCTION

This paper presents a non-linear theory for the equilibrium deformation of homogeneous isotropic shells and makes a comparison between the shell theory and the classical three dimensional non-linear theory of elasticity. These are called the shell theory and the classical theory in the following. In the process estimates are derived which are of considerable interest in their own right.

The classical theory may be obtained by requiring that the potential energy of the shell be stationary with respect to all possible displacements. To obtain the shell theory a restricted set of displacements is admitted into the potential energy as follows. Let θ_1 and θ_2 be parameters for the undeformed middle surface. Let $\vec{r}(\theta_1,\theta_2)$ be the position vector to the undeformed middle surface, $\vec{g}_\alpha = \dfrac{\partial \vec{r}}{\partial \theta_\alpha}(\alpha = 1,2)$ be tangent vectors, and $\vec{g}_3 = \dfrac{\vec{g}_1 \times \vec{g}_2}{|\vec{g}_1 \times \vec{g}_2|}$ be a unit normal vector to the undeformed middle surface. Let θ_3 be the undeformed distance to the middle surface along a normal line, the sign of θ_3 being chosen so that θ_3 is positive on the side of the middle surface towards which $\vec{g}_3$ points and is negative on the other side. Then the displacement vector $\vec{u}$ can be expressed in the form $\vec{u} = U^i \vec{g}_i$ (as usual, Latin indices assume the values 1,2,3; Greek indices assume the values 1,2; and repeated indices are summed). For the shell theory only those displacement vectors are admitted for which U^1 and U^2 are nth degree polynomials in θ_3 and U^3 is an $(n+1)$st degree polynomial in θ_3, the coefficients of the polynomials being functions of θ_1 and θ_2. With this restriction the potential energy becomes a functional of the $3n+4$ coefficients of these polynomials. Requiring that the potential energy be stationary gives the equilibrium equations and surface traction boundary conditions of the shell theory.

The shell theory is compared with the classical theory by examining the errors which result when displacements satisfying the equilibrium equations and surface traction boundary conditions of the shell theory are substituted into the equilibrium equations and surface traction boundary conditions of the classical theory. In doing this it is assumed that the displacements satisfying the shell theory have as many continuous derivatives as desired or needed. After arranging the equilibrium equations and surface traction boundary conditions of the classical theory in a convenient form, $E_{iq} (i = 1,2,3)$ denotes the errors obtained from the equilibrium equations, E_{ie} denotes the errors obtained from the surface traction boundary conditions at the edge, and E_{if} denotes the errors obtained from the surface traction boundary conditions at either face. It is shown that $E_{\alpha q}$, $E_{\alpha e}$, and their derivatives have $n-k-1$ zeros along each line which is normal to the middle surface in the undeformed shell, and E_{3q}, E_{3e}, and their derivatives have $n-k$ zeros along each such line where k is the number of differentiations of the errors with respect to θ_3. Furthermore, if n is large enough, if the shell thickness is small enough relative to the distance to the edge and relative to the geometry of the undeformed shell, if the strains and displacement gradients are small enough after a simplifying rigid transformation, and if the prescribed surface tractions at the faces and their derivatives and the body forces and their derivatives are small enough, then E_{iq}, E_{if}, and their derivatives are significantly small provided the derivatives of the E_{iq} do not have too many differentiations with respect to θ_3. In many cases n is large enough for the above if $n \geq 3$. The meaning of "small enough" depends on n and the strain energy density function. It is also shown that under the previous restrictions the low degree terms in the displacement polynomials and their derivatives are more significant than the high degree terms (this is true at least when the difference in the degrees of the terms is greater than or equal two).

The most difficult task in showing the above is the derivation of estimates for the derivatives of the stresses and displacement gradients. The procedure is to obtain estimates for the L_2 norms for the various functions and use Sobolev's inequality for a slab to obtain pointwise estimates. The work of F. John[1] was an indispensible guide in obtaining

[1] John, F., Estimates for the Derivatives of the Stresses in a Thin Shell and Interior Shell Equations, Comm. Pure Appl. Math., Vol. XVIII, 1965, pp. 235-267.

these estimates (although the details of the calculation are quite different here), and this author acknowledges his indebtedness to that work. The extent of the indebtedness to the work of F. John will be clear to everyone familiar with his work and will not be mentioned further. Appreciation is also hereby expressed to F. John, J. J. Stoker, and W. T. Koiter for valuable discussions on the work.

1. Pseudo-tensor notation.[2]

Consider a fixed rectangular Cartesian reference frame X, and let $\vec{r}(\theta_1,\theta_2)$ be the position vector from the origin of this reference frame to the middle undeformed surface. A middle undeformed surface is considered such that normals to it do not intersect for $|\theta_3| \leq h$, whenever h is small enough, and such that the surface and its boundary are smooth enough so that subsequent uses of the divergence theorem are valid. The undeformed shell is then defined to be the region $|\theta_3| \leq h$. Letting $\vec{x}$ be the position vector from the origin to an arbitrary point (x_1,x_2,x_3) in the shell, we also assume that the middle

[2] Although several of the quantities introduced here have established names such as two point tensors or shifters of one sort or another, the one term "pseudo-tensor" is introduced to keep the nomenclature to a minimum. See Erickson, J. L., Tensor Fields, Handbuch der Physik, Vol. III/1, Springer-Verlag, 1960, pp. 794-850.

undeformed surface is smooth enough so that

$$\vec{x} = \vec{r} + \theta_3 \vec{g}_3 \tag{1.1}$$

defines a relationship between the coordinates x_i and θ_i which has as many continuous derivatives as needed in the following, the same being true for the inverse relationship.

Let

$$g_{ij} = \vec{g}_i \cdot \vec{g}_j \ , \quad (g^{ij}) = (g_{ij})^{-1} \ , \quad \text{and } \vec{g}^i = g^{ij}\vec{g}_j \ . \tag{1.2}$$

Then the quantities $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are the components of the usual covariant and contravariant metric tensors for the middle undeformed surface. Also $g_{3\alpha} = g^{3\alpha} = 0$ and $g_{33} = g^{33} = 1$. Finally $g^{ij} = \vec{g}^i \cdot \vec{g}^j$, $\vec{g}_i \cdot \vec{g}^j = \delta_i^j$, and $\vec{g}_3 = \vec{g}^3$.

The quantities a_j^i and A_j^i are defined to be the components of the vectors $\vec{g}_i$ and $\vec{g}^i$ with respect to the reference frame X as follows:

$$\vec{g}_i = (a_i^1, a_i^2, a_i^3) \ , \quad \vec{g}^i = (A_1^i, A_2^i, A_3^i) \ . \tag{1.3}$$

Then

$$(a_j^i) = (A_j^i)^{-1} \ , \quad g_{ij} = a_i^k a_j^k \ , \quad g^{ij} = A_k^i A_k^j \ . \tag{1.4}$$

Given any indexed set of functions associated with the X-frame, say d_{ij} (the number of indices is not important), we associate indexed sets D_{ij}, $D_i{}^j$, $D^i{}_j$, and D^{ij} with the curvilinear coordinates θ_i as follows:

$$D_{ij} = a_i^k a_j^\ell d_{k\ell} \;, \quad D_i{}^j = a_i^k A_\ell^j d_{k\ell} \;,$$

$$D^i{}_j = A_k^i a_j^\ell d_{k\ell} \;, \quad D^{ij} = A_k^i A_\ell^j d_{k\ell} \;.$$

When the indexed set d_{ij} is symmetric (i.e. $d_{ij} = d_{ji}$), both $D^i{}_j$ and $D_j{}^i$ will be denoted by D_j^i. The functions d_{ij} and D_{ij} (or $D^i{}_j$, $D_i{}^j$, D^{ij}) will be called X- and θ-components of the same pseudo-tensor. It is easily seen that $D_{ij} = g_{ik}D^k{}_j$, $D^{ij} = g^{ik}g^{j\ell}D_{k\ell}$, etc., so that the quantities g^{ij} and g_{ij} can be used to raise and lower indices of the θ-components of pseudo-tensors in the same way that the indices of the components of tensors are raised and lowered using a metric tensor.

Observe that δ_{ij} and g_{ij} (or g^{ij}, δ_j^i) are X- and θ-components of the same pseudo-tensor. Also if u_i denotes the X-components of displacement, then u_i and U^i are X- and θ-components of the same pseudo-tensor.

Pseudo-tensors have a contraction principal of the same sort as ordinary tensors. Let c_{ij} and C_{ij} (or d_{ijk} and D_{ijk}) be X- and θ-components of the same pseudo-tensors. Then $c_{ii} = C_i{}^i = C^i{}_i$ (or d_{iij} and $D_i{}^i{}_j$ are X- and θ-components of the same pseudo-tensor).

If d_{ij} and D_{ij} are X- and θ-components of a pseudo-tensor, $D_{ij}|_k$, $D_{ij}|_{k\ell}$, etc. are defined to be the θ-components of the pseudo-tensors having $\frac{\partial d_{ij}}{\partial x_k}$, $\frac{\partial^2 d_{ij}}{\partial x_k \cdot \partial x_\ell}$, etc. as X-components. i.e. $D_{ij}|_k = a_i^\ell a_j^m a_k^n \frac{\partial d_{\ell m}}{\partial x_n}$, etc. Indices of $D_{ij}|_k$ are raised and lowered the same as those of the θ-components of other pseudo-tensors.

It follows immediately that $g_{ij}|_k = 0$, $g^{ij}|_k = 0$, etc.

Let $b_{\alpha\beta} = \vec{g}_3 \cdot \vec{g}_{\alpha,\beta}$ be the components of the second fundamental form of the middle undeformed surface ($,\alpha$ means $\frac{\partial}{\partial\theta_\alpha}$ and $,i$ will mean $\frac{\partial}{\partial\theta_i}$). For convenience let $b_{i3} = b_{3i} = 0$ and raise the indices of the quantities b_{ij} just as if they were the θ-components of a pseudo-tensor. Then let $(B_j^i) = (\delta_j^i - \theta_3 b_j^i)^{-1}$ (observe that $B_3^\alpha = B_\alpha^3 = 0$, $B_3^3 = 1$). The indices of the B_j^i are also raised and lowered just as if they were θ-components of a pseudo-tensor. Then $H = \frac{1}{2}b_\alpha^\alpha$ is the mean curvature and $K = \det(b_\beta^\alpha)$ is the Gaussian curvature of the middle undeformed surface. For convenience let $C = 1 - 2H\theta_3 + K\theta_3^2$.

Quantities Γ_{ij}^k are defined by $\vec{g}_{i,j} = \Gamma_{ij}^k \vec{g}_k$. Then also $\vec{g}^i{}_{,j} = -\Gamma_{kj}^i \vec{g}^k$. The quantities $\Gamma_{\alpha\beta}^\gamma$ are the usual Christoffel symbols for the metric of the middle undeformed surface and $\Gamma_{ij}^3 = b_{ij}$, $\Gamma_{3j}^i = -b_j^i$, $\Gamma_{j3}^i = 0$.

In terms of the X-components of $\vec{g}_i$ and $\vec{g}^i$ one has

$$(1.4) \qquad a_{i,j}^k = \Gamma_{ij}^\ell a_\ell^k \; ; \quad A_{k,j}^i = -\Gamma_{\ell j}^i A_k^\ell \; .$$

From $\vec{x} = \vec{r} + \theta_3 \vec{g}_3$ one obtains

$$\vec{x}_{,j} = (\delta_j^k - \theta_3 b_j^k)\vec{g}_k$$

so that

$$(1.5) \qquad \begin{cases} \dfrac{\partial x_i}{\partial \theta_j} = a_k^i(\delta_j^k - \theta_3 b_j^k) \ , \\ \\ \dfrac{\partial \theta_i}{\partial x_j} = B_k^i A_j^k \ . \end{cases}$$

If d_i and D_i are X- and θ-components of the same pseudo-tensor,

$$D_i|_j = a_i^k a_j^\ell \frac{\partial d_k}{\partial x_\ell} = a_i^k a_j^\ell \frac{\partial \theta_r}{\partial x_\ell} \frac{\partial}{\partial \theta_r} (A_k^s D_s)$$

$$(1.6) \qquad = a_i^k a_j^\ell B_t^r A_\ell^t (A_k^s D_{s,r} - \Gamma_{ur}^s A_k^u D_s)$$

$$= B_j^r (D_{i,r} - \Gamma_{ir}^s D_s) \ .$$

Similarly

$$(1.7) \qquad D^i|_j = (D^i{}_{,r} + \Gamma_{sr}^i D^s) B_j^r \ .$$

If d_{ij} and D_{ij} are X- and θ-components of the same pseudo-tensor,

$$D^{ij}|_k = (D^{ij}{}_{,\ell} + \Gamma^i_{m\ell} D^{mj} + \Gamma^j_{m\ell} D^{im}) B^\ell_k ,$$

$$D^i{}_j|_k = (D^i{}_{j,\ell} + \Gamma^i_{m\ell} D^m{}_j - \Gamma^m_{j\ell} D^i{}_m) B^\ell_k ,$$

(1.8)

$$D_{ij}|_k = (D_{ij,\ell} - \Gamma^m_{i\ell} D_{mj} - \Gamma^m_{j\ell} D_{im}) B^\ell_k ,$$

$$D^i{}_j|^k = (D^i{}_{j,\ell} + \Gamma^i_{m\ell} D^m{}_j - \Gamma^m_{j\ell} D^i{}_m) B^{\ell k} , \text{ etc.}$$

2. Introduction of the classical theory and shell theory.

Let u_i denote the X-components of displacement, $e_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j})$ be the strains, and let $s_1 = e_{ii}$, $s_2 = e_{ij}e_{ji}$, and $s_3 = e_{ij}e_{jk}e_{ki}$ be the strain invariants.

For homogeneous isotropic materials, the strain energy per unit undeformed volume, W, is a function of s_1, s_2, and s_3. In order that the strain energy density function agrees to lowest order terms with that of the linear theory of elasticity, it follows from Taylor's theorem that

$$W = \frac{\lambda}{2} s_1^2 + \mu s_2 + W_1 s_3 + W_2 s_1 s_2 + W_3 s_2^2 + W_4 s_1^3$$

where λ and μ are the Lamé constants and W_1, W_2, W_3, and W_4 are functions of s_1, s_2, and s_3 which are assumed to have as many derivatives as necessary for the following.

Let

$$t_{ij} = \frac{\partial W}{\partial s_1}\delta_{ij} + 2\frac{\partial W}{\partial s_2}e_{ij} + 3\frac{\partial W}{\partial s_3}e_{ik}e_{kj}$$

so that

$$t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + z_{ij}$$

where z_{ij} is quadratic in the $e_{k\ell}$ with coefficients depending on the $e_{k\ell}$.

Let

$$q_{ij} = (\delta_k^i + \frac{\partial u_i}{\partial x_k})t_{kj} = \frac{\partial W}{\partial(\frac{\partial u_i}{\partial x_j})}$$

be the Kirchhoff stresses. The q_{ij} are always treated as functions of the displacement gradients and are defined for all displacements even if the displacements are not solutions of either the shell theory or the classical theory.

Let $\overline{q}_{i+}$ be the X-components of surface traction (force per unit undeformed area) on the face $\theta_3 = h$, and let $\overline{q}_{i-}$ be the corresponding components on the face $\theta_3 = -h$.

Let f_i be the X-components of body force (force per unit undeformed volume).

Let $\overline{q}_{ie}$ be the X-components of surface traction (force per unit undeformed area) at the edge of the shell.

Then the equilibrium equations of the classical theory are

$$(2.1) \qquad \frac{\partial q_{ij}}{\partial x_j} + f_i = 0$$

and at the boundary

$$(2.2) \qquad \begin{cases} \bar{q}_{i+} = q_{ij}n_j & \text{for} \quad \theta_3 = h \\ \bar{q}_{i-} = q_{ij}n_j & \text{for} \quad \theta_3 = -h \\ \bar{q}_{ie} = q_{ij}n_j & \text{at the edge} \end{cases}$$

where the n_i are the X-components of the unit outer normal vector to the shell.

Now let capital letters denote the θ-components of pseudo-tensors whose X-components are denoted by the lower case letters. Then, using the contraction principal for pseudo-tensors, the classical theory becomes

$$(2.3) \qquad \begin{cases} Q^{ij}\big|_j + F^i = 0 \; , \\ \bar{Q}^i{}_+ = Q^{i3} & \text{for} \quad \theta_3 = h \\ \bar{Q}^i{}_- = -Q^{i3} & \text{for} \quad \theta_3 = -h \\ \bar{Q}^i{}_e = Q^{i\alpha}N_\alpha & \text{at the edge} \end{cases}$$

since $N_\alpha = 0$, $N_3 = \pm 1$ for $\theta_3 = \pm h$ and $N_3 = 0$ on the edge.

To obtain the shell theory, consider the potential energy

$$\mathcal{E} = \iiint\limits_{\text{shell}} (W - f_i u_i)dv - \iint\limits_{\theta_3=h} \bar{q}_{i+}u_i dS - \iint\limits_{\theta_3=-h} \bar{q}_{i-}u_i dS - \iint\limits_{\text{edge}} \bar{q}_{ie}u_i dS$$

where dv is the undeformed volume element and dS is the undeformed area element. Letting a supper dot denote the variation, the requirement that $\mathcal{E}$ is stationary becomes

$$\dot{\mathcal{E}} = \iiint\limits_{\text{shell}} (q_{ij} \frac{\partial \dot{u}_i}{\partial x_j} - f_i \dot{u}_i) dv - \iint\limits_{\theta_3 = h} \overline{q}_{i+} \dot{u}_i dS$$

$$- \iint\limits_{\theta_3 = -h} \overline{q}_{i-} \dot{u}_i dS - \iint\limits_{\text{edge}} \overline{q}_{ie} \dot{u}_i dS = 0 \ .$$

After using the divergence theorem, this becomes

$$0 = -\iiint\limits_{\text{shell}} (\frac{\partial q_{ij}}{\partial x_j} + f_i) \dot{u}_i dv + \iint\limits_{\theta_3 = h} (q_{ij} n_j - \overline{q}_{i+}) \dot{u}_i dS$$

$$+ \iint\limits_{\theta_3 = -h} (q_{ij} n_j - \overline{q}_{i-}) \dot{u}_i dS + \iint\limits_{\text{edge}} (q_{ij} n_j - \overline{q}_{ie}) \dot{u}_i dS \ .$$

Using the contraction principal and the fact that the $\dot{U}_i$ are the θ-components of the pseudo-tensor whose X-components are the $\dot{u}_i$, one has

$$0 = -\iiint\limits_{\text{shell}} (Q^{ij}|_j + F^i) \dot{U}_i dv + \iint\limits_{\theta_3 = h} (Q^{ij} N_j - \overline{Q}^i{}_+) \dot{U}_i dS$$

$$+ \iint\limits_{\theta_3 = -h} (Q^{ij} N_j - \overline{Q}^i{}_-) \dot{U}_i dS + \iint\limits_{\text{edge}} (Q^{ij} N_j - \overline{Q}^i{}_e) \dot{U}_i dS \ .$$

Since $N_\alpha = 0$, $N_3 = \pm 1$ for $\theta_3 = \pm h$ and $N_3 = 0$ on the edge, the above becomes

$$0 = -\iiint\limits_{\text{shell}} (Q^{ij}|_j + F^i)\dot{U}_i dv + \iint\limits_{\theta_3 = h} (Q^{i3} - \overline{Q}^i{}_+)\dot{U}_i dS$$

$$- \iint\limits_{\theta_3 = -h} (Q^{i3} + \overline{Q}^i{}_-)\dot{U}_i dS + \iint\limits_{\text{edge}} (Q^{i\alpha} N_\alpha - \overline{Q}^i{}_e)\dot{U}_i dS .$$

But $dv = CdAd\theta_3$ and $dS = CdA$ for $\theta_3 = \pm h$ where dA is the element of undeformed area on the middle surface. Also $dS = \mathcal{Q} ds\, d\theta_3$ at the edge where $\mathcal{Q} = \sqrt{(\delta^\alpha_\beta - 2\theta_3 b^\alpha_\beta + \theta_3^2 b^\alpha_\gamma b^\gamma_\beta)\lambda^\beta \lambda_\gamma}$, $\lambda^\alpha \vec{g}_\alpha = \lambda_\alpha \vec{g}^\alpha$ is the unit tangent vector to the boundary curve on the undeformed middle surface, and s is the undeformed arc length of the boundary curve on the middle surface. In addition, for the displacements admitted into the potential energy to obtain the shell theory,

$$\dot{U}_\alpha = \sum_{k=0}^{n} \dot{U}_{\alpha k} \theta_3{}^k , \quad \dot{U}_3 = \sum_{k=0}^{n+1} \dot{U}_{3k} \theta_3{}^k$$

where $\dot{U}_{\alpha k}$ and $\dot{U}_{3k}$ are arbitrary functions of θ_1 and θ_2. Hence

$$0 = \iint \sum_{k=0}^{n} \left\{ -\int_{-h}^{h} (Q^{\alpha j}|_j + F^{\alpha}) c\theta_3^k d\theta_3 \right.$$

$$+ [(Q^{\alpha 3} - \overline{Q}^{\alpha}_{+}) c\theta_3^k]_{\theta_3=h} - [(Q^{\alpha 3} + \overline{Q}^{\alpha}_{-}) c\theta_3^k]_{\theta_3=-h} \Big\} \dot{U}_{\alpha k}\, dA$$

$$+ \iint \sum_{k=0}^{n+1} \left\{ -\int_{-h}^{h} (Q^{3j}|_j + F^{3}) c\theta_3^k d\theta_3 \right.$$

$$+ [(Q^{33} - \overline{Q}^{3}_{+}) c\theta_3^k]_{\theta_3=h} - [(Q^{33} + \overline{Q}^{3}_{-}) c\theta_3^k]_{\theta_3=-h} \Big\} \dot{U}_{3k}\, dA$$

$$+ \int \sum_{k=0}^{n} \left\{ \int_{-h}^{h} Q^{\beta\alpha} N_\alpha - \overline{Q}^{\beta}_{e}) \theta_3^k a\, d\theta_3 \right\} \dot{U}_{\beta k}\, ds$$

$$+ \int \sum_{k=0}^{n+1} \left\{ \int_{-h}^{h} Q^{3\alpha} N_\alpha - \overline{Q}^{3}_{e}) \theta_3^k a\, d\theta_3 \right\} \dot{U}_{3k}\, ds \; .$$

From this are obtained the equilibrium equations

$$(2.4) \begin{cases} \displaystyle\int_{-h}^{h} (Q^{\alpha j}|_j + F^{\alpha}) c\theta_3^k d\theta_3 = [(Q^{\alpha 3} - \overline{Q}^{\alpha}_{+}) c\theta_3^k]_{\theta_3=h} \\ \quad - [(Q^{\alpha 3} + \overline{Q}^{\alpha}_{-}) c\theta_3^k]_{\theta_3=-h} \quad (k = 0,1,2,\ldots,n) \\ \displaystyle\int_{-h}^{h} (Q^{3j}|_j + F^{3}) c\theta_3^k d\theta_3 = [(Q^{33} - \overline{Q}^{3}_{+}) c\theta_3^k]_{\theta_3=h} \\ \quad - [(Q^{33} + \overline{Q}^{3}_{-}) c\theta_3^k]_{\theta_3=-h} \quad (k = 0,1,2,\ldots,n+1) \end{cases}$$

and the edge boundary conditions

$$
(2.5)\quad \begin{cases} \displaystyle\int_{-h}^{h} (Q^{\alpha\beta}N_\beta - \overline{Q}^{\alpha}{}_e)\theta_3^k\,\mathcal{U}\,d\theta_3 = 0\,(k = 0,1,2,\ldots,n)\ , \\ \displaystyle\int_{-h}^{h} (Q^{3\beta}N_\beta - \overline{Q}^{3}{}_e)\theta_3^k\,\mathcal{U}\,d\theta_3 = 0\,(k = 0,1,2,\ldots,n+1) \end{cases}
$$

of the shell theory.

Letting

$$
(2.6)\qquad \overline{Q}^i = \frac{\theta_3}{2h}(\overline{Q}^i{}_+ + \overline{Q}^i{}_-) + \frac{1}{2}(\overline{Q}^i{}_+ - \overline{Q}^i{}_-)
$$

one has $\overline{Q}^i = \overline{Q}^i{}_+$ when $\theta_3 = h$ and $\overline{Q}^i = -\overline{Q}^i{}_-$ when $\theta_3 = -h$.

Then the equilibrium equations become

$$
(2.7)\quad \begin{cases} \displaystyle\int_{-h}^{h} (Q^{\alpha j}|_j + F^{\alpha})C\theta_3^k d\theta_3 = [(Q^{\alpha 3} - \overline{Q}^{\alpha})C\theta_3^k]_{\theta_3=-h}^{\theta_3=h}\ ,\ (k = 0,1,2,\ldots,n) \\ \displaystyle\int_{-h}^{h} (Q^{3j}|_j + F^{3})C\theta_3^k d\theta_3 = [(Q^{33} - \overline{Q}^{3})C\theta_3^k]_{\theta_3=-h}^{\theta_3=h}\ ,\ (k = 0,1,2,\ldots,n+1). \end{cases}
$$

From these equilibrium equations it follows that

$$
(2.8)\quad \begin{cases} \displaystyle\int_{-h}^{h} (Q^{ij}|_j + F^{i})CP\,d\theta_3 = [(Q^{i3} - \overline{Q}^{i})CP]_{\theta_3=-h}^{\theta_3=h} \\ \displaystyle\int_{-h}^{h} [Q^{ij}|_j + F^{i})C]_{,\alpha}P\,d\theta_3 = \Big\{[(Q^{i3} - \overline{Q}^{i})C]_{,\alpha}P\Big\}_{\theta_3=-h}^{\theta_3=h}\ , \end{cases}
$$

etc. where P is an arbitrary polynomial in θ_3 of degree n if $i \neq 3$ and of degree $n+1$ if $i = 3$.

Similarly

$$(2.9)\quad \begin{cases} \displaystyle\int_{-h}^{h} (Q^{i\beta}N_\beta - \overline{Q}^i{}_e)\,\mathcal{A}\,P\,d\theta_3 = 0 \\ \displaystyle\int_{-h}^{h} [(Q^{i\beta}N_\beta - \overline{Q}^i{}_e)\mathcal{A}]_{,s}\,P\,d\theta_3 = 0 \end{cases}$$

etc. at the edge of the shell where P has the same meaning as above.

3. General comparison of the shell theory and classical theory.

This comparison of the two theories is called a general comparison because it is valid for all solutions to the shell theory, regardless of the thickness of the shell or the size of the deformation. This comparison is contained in the following theorem.

Theorem (3.1): Let the displacements U^i be a solution to equilibrium equations and surface traction boundary conditions of the shell theory. Then

$$\frac{\partial^k[(Q^{\alpha j}|_j + F^\alpha)C]}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}} = 0 \text{ at } n - k_3 - 1 \text{ points } (\text{if } n - k_3 - 1 > 0), \text{ and}$$

$$\frac{\partial^k[(Q^{3j}|_j + F^3)C]}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}} = 0 \text{ at } n - k_3 \text{ points } (\text{if } n - k_3 > 0)$$

along each line which is normal to the middle surface in the undeformed shell. Also at the edge of the shell

$$\frac{\partial^k[(Q^{\alpha\beta}N_\beta - \overline{Q}^\alpha{}_e)\mathcal{A}]}{\partial s^{k_1}\partial\theta_3^{k_2}} = 0 \text{ at } n-k_2-1 \text{ points (if } n-k_2-1 > 0\text{), and}$$

$$\frac{\partial^k[(Q^{3\beta}N_\beta - \overline{Q}^3{}_e)\ell]}{\partial s_1^{k_1}\partial\theta_3^{k_2}} = 0 \text{ at } n-k_2 \text{ points (if } n-k_2 > 0\text{)}$$

along each line which is normal to the undeformed middle surface.

The equilibrium equations and edge boundary conditions of the classical theory may be written as

$$(Q^{ij}|_j + F^i)C = 0$$

$$(Q^{i\beta}N_\beta - \overline{Q}^i{}_e)\mathcal{A} = 0 \ .$$

Thus the theorem states that if a solution to the shell theory is substituted into these equations, both the error and its derivatives are zero at a stated number of points on each normal line to the middle surface.

The proof of the statement about $\dfrac{\partial^k[(Q^{3j}|_j + F^3)C]}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}}$ will be given. The rest of the theorem can be proved in a similar manner. Let $E = \dfrac{\partial^{k-k_3}[(Q^{3j}|_j + F^3)C]}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}}$. For each fixed θ_1, θ_2 it is assumed that E has isolated zeros. Otherwise it has infinitely many and the statement of the theorem for E is trivially true.

First consider the case $n \geq 1$ and let $P = h^2 - \theta_3^2$. From (2.8) one has $\int_{-h}^{h} EPd\theta_3 = 0$. Since P is not zero for $|\theta_3| < h$, it follows that E has at least one zero for $|\theta_3| < h$. Furthermore, since its zeros are isolated, it has at least one zero where it changes sign.

Now consider the case $n \geq 2$. Assume E has only one zero where it changes sign, namely $\theta_3 = r$. Let $P = (\theta_3 - r)(h^2 - \theta_3^2)$. Again $\int_{-h}^{h} EPd\theta_3 = 0$. From the choice of P, the integrand does not change sign. This is a contradiction so that E has at least two zeros where it changes sign.

Proceeding in this manner, it can be shown that E has n zeros for $|\theta_3| < h$. Then using the mean value theorem, the statement of the theorem about $\dfrac{\partial^k[(Q^{3j}|_j + F^3)C]}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}}$ follows.

4. Certain dependencies.

In deriving estimates for the derivatives of the displacement gradients and stresses, use will be made of certain dependencies which exist among them.

Let $L_{ij} = \frac{1}{2}(U^i{}_{,j} + U^j{}_{,i})$ and $M_{ij} = \lambda L_{kk}\delta_{ij} + 2\mu L_{ij}$. Then the L_{ij} and M_{ij} would be the strains and stresses if a flat plate and the linear theory of elasticity were being considered with $\theta_\alpha = x_\alpha$.

Let $S_{ij} = Q^i{}_j - M_{ij}$ and $V_i = M_{ij,j}$. Then S_{ij} is the part of the stress which is due to non-linearity and the curvature of the shell. Also V_i would be identically zero if a flat plate and the linear theory were being considered with zero body forces and with $\theta_\alpha = x_\alpha$.

Let $a_k (k = 1,2,\ldots)$ be the set of all functions of the types

$$\frac{\partial^k U^i}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}} \; , \quad \frac{\partial^{k-1} M_{j\ell}}{\partial\theta_1^{\ell_1}\partial\theta_2^{\ell_2}\partial\theta_3^{\ell_3}} \; , \quad \frac{\partial^{k-1} Q^j{}_\ell}{\partial\theta_1^{\ell_1}\partial\theta_2^{\ell_2}\partial\theta_3^{\ell_3}}$$

where for functions of the first type $i \neq 3$ and k_3 is even, or $i = 3$ and k_3 is odd; for functions of the second and third types $\ell_3 = 0$, or $\ell_3 = 1$ and at least one of j and ℓ is 3, or ℓ_3 is even and neither or both of j and ℓ are 3, or ℓ_3 is odd and exactly one of j and ℓ are 3. The various cases mentioned are not all mutually exclusive.

Let $b_k (k = 1,2,\ldots)$ be the set of all functions of the above three types with no restrictions on the indices.

Theorem (4.1): <u>For</u> $k \geq 2$ <u>every function in</u> a_k <u>can be expressed as a linear combination of functions of the types</u>

$$\frac{\partial^{k-1} M_{ij}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}} \; , \quad \frac{\partial^{k-2} V_\ell}{\partial\theta_1^{\ell_1}\partial\theta_2^{\ell_2}\partial\theta_3^{\ell_3}} \; , \quad \frac{\partial^{k-1} S_{rs}}{\partial\theta_1^{r_1}\partial\theta_2^{r_2}\partial\theta_3^{r_3}}$$

where $\ell_3 = 0$, or ℓ_3 is even and $\ell \neq 3$, or ℓ_3 is odd and $\ell_3 = 3$.

For $k \geq 2$ every function in b_k can be expressed as a linear combination of functions of the types

$$\frac{\partial^{k-1} M_{ij}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}}, \quad \frac{\partial^{k-1} M_{\alpha\beta}}{\partial\theta_1^{\ell_1}\partial\theta_2^{\ell_2}\partial\theta_3}, \quad \frac{\partial^{k-2} V_m}{\partial\theta_1^{m_1}\partial\theta_2^{m_2}\partial\theta_3^{m_3}}, \quad \frac{\partial^{k-1} S_{rs}}{\partial\theta_1^{r_1}\partial\theta_2^{r_2}\partial\theta_3^{r_3}} .$$

These linear combinations for functions in a_k and b_k can be chosen so that the following are true. The derivatives of S_{rs} occur only in the linear combinations for the derivatives of the $Q^i{}_j$. The linear combinations for $\dfrac{\partial^{k-1} M_{ij}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}}$ $(k_3 \geq 1)$ involve derivatives of V_ℓ which have at most $k_3 - 1$ differentiations with respect to θ_3. The linear combinations for $U^3{}_{,3\alpha}$ do not involve V_ℓ or any of its derivatives. Finally, the linear combinations for $\dfrac{\partial^k U^i}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}}$ involve derivatives of V_ℓ which have at most $k_3 - 2$ differentiations with respect to θ_3 if $k_3 \geq 2$, and no derivatives of V_ℓ with respect to θ_3 are involved if $k_3 < 2$.

All linear combinations referred to have constant coefficients which depend only on λ and μ.

To prove the theorem use is made of the relations

$$(4.2)\quad U^i{}_{,jk} = \frac{1}{2\mu}\,[M_{ij,k} + M_{ik,j} - M_{jk,i} - \frac{\lambda}{3\lambda+2\mu}\,(M_{\ell\ell,k}\delta_{ij} + M_{\ell\ell,j}\delta_{ik} - M_{\ell\ell,i}\delta_{jk})]$$

and

$$(4.3)\quad M_{ij} = \lambda U^k{}_{,k}\delta_{ij} + \mu(U^i{}_{,j} + U^j{}_{,i}) \ .$$

From the definitions of L_{ij} and M_{ij}

$$U^i{}_{,jk} = L_{ij,k} + L_{ik,j} - L_{jk,i} \ ,$$

$$L_{ij} = \frac{1}{2\mu}\,(M_{ij} - \frac{\lambda}{3\lambda+2\mu}\,M_{kk}\delta_{ij})$$

and (4.2) follows from these.

Eq. (4.3) also follows from the definitions of M_{ij} and L_{ij}.

From the definition of V_i

$$(4.4)\quad M_{i3,3} = -\,M_{i\alpha,\alpha} + V_i \ .$$

From (4.2)

$$(4.5)\quad \begin{cases} U^\alpha{}_{,\beta\gamma} = \frac{1}{2\mu}\,[M_{\alpha\beta,\gamma} + M_{\alpha\gamma,\beta} - M_{\beta\gamma,\alpha} - \frac{\lambda}{3\lambda+2\mu}\,(M_{\ell\ell,\gamma}\delta_{\alpha\beta} + M_{\ell\ell,\beta}\delta_{\alpha\gamma} \\ \qquad\qquad -M_{\ell\ell,\alpha}\delta_{\beta\gamma})] \\ U^\alpha{}_{,\beta 3} = \frac{1}{2\mu}\,[M_{\alpha\beta,3} + M_{\alpha 3,\beta} - M_{\beta 3,\alpha} - \frac{\lambda}{3\lambda+2\mu}\,M_{\ell\ell,3}\delta_{\alpha\beta}] \\ U^\alpha{}_{,33} = \frac{1}{2\mu}\,[2M_{\alpha 3,3} - M_{33,\alpha} + \frac{\lambda}{3\lambda+2\mu}\,M_{\ell\ell,\alpha}] \\ U^3{}_{,\alpha\beta} = \frac{1}{2\mu}\,[M_{3\alpha,\beta} + M_{3\beta,\alpha} - M_{\alpha\beta,3} + \frac{\lambda}{3\lambda+2\mu}\,M_{\ell\ell,3}\delta_{\alpha\beta}] \\ U^3{}_{,3\alpha} = \frac{1}{2\mu}\,[M_{33,\alpha} - \frac{\lambda}{3\lambda+2\mu}\,M_{\ell\ell,\alpha}] \\ U^3{}_{,33} = \frac{1}{2\mu}\,[M_{33,3} - \frac{\lambda}{3\lambda+2\mu}\,M_{\ell\ell,3}] \end{cases}$$

The statement of the theorem about the derivatives of M_{ij} in a_2 and b_2 is seen to be true from (4.4). Substituting (4.4) into (4.5), the statement of the theorem about the derivatives of U^i in a_2 and b_2 is seen to be true. The statement of the theorem about the derivatives of $Q^i{}_j$ in a_2 and b_2 follow using the definition of S_{ij} and the statement of the theorem about the derivatives of M_{ij}.

Differentiating (4.4) and (4.5) with respect to θ_1 and θ_2, the statement of the theorem is obtained for those derivatives of M_{ij} and U^i in a_3 and b_3 in which not all differentiations are with respect to θ_3. The statement about the rest of the derivatives of M_{ij} and U^i in a_3 and b_3 follow from the preceding and

$$
(4.6)\quad \left\{
\begin{array}{ll}
\left.\begin{array}{l}
M_{\alpha 3,33} = -M_{\alpha\beta,\beta 3} + V_{\alpha,3} \\[1ex]
M_{33,33} = -M_{3\alpha,\alpha 3} + V_{3,3}
\end{array}\right\} & \text{from (4.4)} \\[3ex]
\left.\begin{array}{l}
U^3{}_{,333} = \frac{1}{\lambda+2\mu}\,(M_{33,33} - \lambda U^\gamma{}_{,\gamma 33}) \\[1ex]
M_{\alpha\beta,33} = \lambda U^k{}_{,k33}\delta_{\alpha\beta} + \mu(U^\alpha{}_{,\beta 33} + U^\beta{}_{,\alpha 33})
\end{array}\right\} & \text{from (4.3)} \\[3ex]
U^\alpha{}_{,333} = \frac{1}{2\mu}\,[2M_{\alpha 3,33} - M_{33,\alpha 3} + \frac{\lambda}{3\lambda+2\mu}\, M_{\ell\ell,\alpha 3}] & \text{from (4.5)}
\end{array}
\right.
$$

The statement of the theorem about the derivatives of $Q^i{}_j$ in a_3 and b_3 follow again from the definition of S_{ij} and from the statement about the derivatives of M_{ij} in a_3 and b_3.

The statement of the theorem is proved for functions in a_4 and b_4 exactly as for those in a_3 and b_3, etc.

5. The parameters used in making estimates, miscellaneous notation and results.

After picking any point on the middle undeformed surface away from the edge, the X-axes can be introduced so that the chosen point will be at the origin and the equation of the middle undeformed surface will have the form $x_3 = f(x_1,x_2)$ in some neighborhood of the origin with $f(0,0) = 0$ and $\dfrac{\partial f(0,0)}{\partial x_\alpha} = 0$. After an appropriate rigid transformation of the deformed shell (moving surface tractions and body forces with the deformed shell), one also has $u_i(0,0,0) = 0$, and $\dfrac{\partial u_i}{\partial x_j} = \dfrac{\partial u_j}{\partial x_i}$ at $\vec{x} = (0,0,0)$.

Let $\theta_\alpha = x_\alpha$ for the remainder of the paper and choose D so that it is less than or equal the distance from the X_3-axis to the boundary of the middle undeformed surface and so that $\det(g_{ij}) \geq \frac{1}{2}$ for $\theta_1^2 + \theta_2^2 \leq D^2$ (observe that $\det(g_{ij}) = 1$ at the origin).

Choose R so that $|f_{,\alpha\beta}| \leq \frac{1}{R}$, $|f_{,\alpha\beta\gamma}| \leq \frac{1}{R^2}$, etc. for $\theta_1^2 + \theta_2^2 \leq D^2$ and for as high an order of derivatives as needed in the following.

Restrict h so that $C \geq \frac{1}{2}$ for $|\theta_3| \leq h$ and $\theta_1^2 + \theta_2^2 \leq D^2$.

Consider shell thicknesses, surface tractions, body forces, and displacements which satisfy the equilibrium equations of the shell theory such that

$$
(5.1)\begin{cases}
\dfrac{h}{D} \le \theta_0 \; , \; \dfrac{h}{R} \le \theta_0^2 \; , \quad \left|\dfrac{\partial u_i}{\partial x_j}\right| \le \theta_0^2 \; , \quad |e_{ij}| \le \theta_0^2 \; , \\[2ex]
\left|\dfrac{\partial^k \overline{Q}^i_{\pm}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}}\right| \le \dfrac{Y\theta_0^2}{h^k} \; , \quad \left|\dfrac{\partial^k F^i}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}}\right| \le \dfrac{Y\theta_0^2}{h^{k+1}} \text{ for } \theta_1^2+\theta_2^2 \le D^2 \text{ and}
\end{cases}
$$

$|\theta_3| \le h$ and $k = 0,1,2,\ldots$ up to the highest order required in the following where Y is Young's modulus and θ_0 is a constant which satisfies $0 < \theta_0 < 1$, which is small enough for the following to be valid, and which depends only on the strain energy W and n.

Let $\theta = \max\left(\frac{h}{D}, \sqrt{\frac{h}{R}}, \sqrt{\varepsilon}\right)$ so that θ is a function of the arguments $\frac{h}{D}, \frac{h}{R}, \varepsilon$. For $\frac{h}{D}$ and $\frac{h}{R}$ fixed, θ and $\frac{\varepsilon}{\theta}$ are non-decreasing as ε increases. Let ε be the smallest number such that

$$
(5.2)\begin{cases}
\left|\dfrac{\partial u_i}{\partial x_j}\right| \le \dfrac{\theta_0}{\theta}\varepsilon \; , \quad |e_{ij}| \le \varepsilon \; , \quad \left|\dfrac{\partial^k \overline{Q}^\alpha_{\pm}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}}\right| \le \dfrac{Y\theta^{k+1}}{\theta_0^{k+1}h^k}\varepsilon \; , \\[2ex]
\left|\dfrac{\partial^k \overline{Q}^3_{\pm}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}}\right| \le \dfrac{Y\theta^k}{\theta_0^k h^k}\varepsilon \; , \quad \left|\dfrac{\partial^k (\overline{Q}^3_{+} + \overline{Q}^3_{-})}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}}\right| \le \dfrac{Y\theta^{k+2}}{\theta_0^{k+2}h^k}\varepsilon \; , \\[2ex]
\left|\dfrac{\partial^k F^i}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}}\right| \le \begin{cases}
\dfrac{Y\theta^{k+1}}{\theta_0^{k+1}h^{k+1}}\varepsilon & \text{if } k_3 \text{ is even and } i \ne 3, \\ & \text{or } k_3 \text{ is odd and } i = 3 \\[1ex]
\dfrac{Y\theta^{k+2}}{\theta_0^{k+2}h^{k+1}}\varepsilon & \text{if } k_3 = 0 \text{ and } i = 3 \\[1ex]
\dfrac{Y\theta^k}{\theta_0^k h^{k+1}}\varepsilon & \text{otherwise}
\end{cases}
\end{cases}
$$

where $k = 0,1,2,\ldots$ up to the highest order required in the following.

From (5.1) it follows that $\sqrt{\varepsilon} \le \theta_0$ and hence $\theta \le \theta_0$. Let $d = \frac{\theta_0}{\theta} h$. Then listing some results

$$
(5.3) \qquad
\begin{cases}
\frac{d}{h}\theta = \theta_0 \\
\theta \le \frac{h}{d} \\
\varepsilon \le \theta^2 \le \frac{h^2}{d^2} \\
h \le d \le \theta_0 D \\
\frac{h}{R} \le \frac{d}{R} \le \theta_0 \theta \\
\frac{1}{R} \le \frac{\theta_0 \theta}{d}
\end{cases}
$$

From (5.2) and the definition of $\bar{Q}^i$,

$$
(5.4) \qquad
\begin{cases}
\left|\frac{\partial u_i}{\partial x_j}\right| \le \frac{d}{h}\varepsilon , \quad |e_{ij}| \le \varepsilon , \quad \left|\frac{\partial^k \bar{Q}^\alpha}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}}\right| = \mathcal{O}\left(\frac{h\varepsilon}{d^{k+1}}\right), \\
\left|\frac{\partial^k \bar{Q}^\alpha}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3}\right| = \mathcal{O}\left(\frac{\varepsilon}{d^k}\right), \quad \left|\frac{\partial^k \bar{Q}^3}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}}\right| = \mathcal{O}\left(\frac{\varepsilon}{d^k}\right), \\
\qquad \left|\frac{\partial^k \bar{Q}^3}{\partial\theta^{k_1}\partial\theta^{k_2}\partial\theta_3}\right| = \mathcal{O}\left(\frac{h\varepsilon}{d^{k+1}}\right), \\
\left|\frac{\partial^k F^i}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}}\right| = \mathcal{O}
\begin{cases}
\frac{\varepsilon}{d^{k+1}} & \text{if } k_3 \text{ is even and } i \ne 3, \text{ or } k_3 \text{ is odd and } i = 3 \\
\frac{h\varepsilon}{d^{k+2}} & \text{if } k_3 = 0 \text{ and } i = 3 \\
\frac{\varepsilon}{hd^k} & \text{otherwise}
\end{cases}
\end{cases}
$$

for $k = 0,1,2,\ldots$ to the highest order needed, and for $\theta_1^2 + \theta_2^2 \le D^2$ and $|\theta_3| \le h$. Here and in the following, the notation $A = \mathcal{O}(B)$ means $A \ge 0$ and $B \ge 0$ and there is a constant k depending at most on n and W such that $A \le kB$ in the domain under consideration. The constant k is allowed to be dimensional for convenience.

If the left hand sides of inequalities (5.1) are much less than the right hand sides, then $\sqrt{\varepsilon}$ will be much less than θ_0 and θ will be much less than θ_0. Hence both ε and $\frac{h}{d}$ will be small. Estimates for various quantities will be given in terms of ε, h, and d.

Let $p_{ij} = \frac{\partial u_i}{\partial x_k} t_{kj}$ so that $q_{ij} = t_{ij} + p_{ij}$. Let E^i_j, T^i_j, Z^i_j, and $P^i{}_j$ be θ-components of pseudo-tensors whose X-components are e_{ij}, t_{ij}, z_{ij}, and p_{ij} respectively. Also let $R_{ij} = E^i_j - L_{ij}$.

Then using the contraction principal for pseudo-tensors,

$$(5.5)\quad \left\{ \begin{aligned}
E^i_j &= \tfrac{1}{2}(U^i|_j + U_j|^i + U_k|^i U^k|_j) \\
R_{ij} &= \tfrac{1}{2}(U^i|_j + U_j|^i + U_k|^i U^k|_j - U^i{}_{,j} - U^j{}_{,i}) \\
s_1 &= E^i_i \,, \quad s_2 = E^i_j E^j_i \,, \quad s_3 = E^i_j E^j_k E^k_i \\
T^i_j &= \frac{\partial w}{\partial s_1}\delta^i_j + 2\frac{\partial w}{\partial s_2}E^i_j + 3\frac{\partial w}{\partial s_3}E^i_k E^k_j \\
&= \lambda E^k_k \delta^i_j + 2\mu E^i_j + Z^i_j \\
&= M_{ij} + \lambda R_{kk}\delta_{ij} + 2\mu R_{ij} + Z^i_j \\
P^i{}_j &= U^i|_k T^k_j \\
Q^i{}_j &= T^i_j + P^i{}_j \\
S_{ij} &= Q^i{}_j - M_{ij} = \lambda R_{kk}\delta_{ij} + 2\mu R_{ij} + Z^i_j + P^i{}_j \,.
\end{aligned} \right.$$

From the definition of z_{ij}, one has

$$z_{ij} = W_5\delta_{ij} + W_6 e_{ij} + W_7 e_{ik}e_{kj}$$

where W_5, W_6, and W_7 depend only on s_1, s_2, and s_3, and W_5 is quadratic in the $e_{k\ell}$ while W_6 is linear in the $e_{k\ell}$. Hence

$$Z^i_j = W_5\delta^i_j + W_6 E^i_j + W_7 E^i_k E^k_j \ ,$$

and, in view of s_i as given in (5.5), it follows that Z^i_j is quadratic in the E^k_ℓ with coefficients depending on the E^k_ℓ.

When the $|E^i_j|$ are small enough, the equations for the T^i_j can be inverted giving E^i_j as a function which is linear in the T^k_ℓ with coefficients depending on the T^k_ℓ. Therefore Z^i_j is also quadratic in the T^k_ℓ with coefficients depending on the T^k_ℓ if the $|E^i_j|$ are small enough.

For $i = 0,1,2,\ldots$ let G_i denote any function of θ_1, θ_2, θ_3 such that

$$(5.6)\qquad \begin{cases} |G_i| = \mathcal{O}\left(\dfrac{\theta_0\theta}{d^i}\right) \\[2ex] \left|\dfrac{\partial^k G_i}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}}\right| = \mathcal{O}\left(\dfrac{\theta_0\theta}{d^{k+i}}\right) \end{cases}$$

for $k = 0,1,2,\ldots$ to the highest order needed and for θ_1, θ_2, θ_3 in the domain being considered.

Then from (5.3)

$$(5.7)\quad \begin{cases} G_iG_j = G_{i+j}\ , \quad G_i + G_i = G_i\ , \quad \dfrac{\partial G_i}{\partial \theta_j} = G_{i+1}\ , \\[2ex] \theta_3 G_1 = G_0 \text{ and } \theta_3^2 G_2 = G_0 \text{ for } |\theta_3| \le h\ . \end{cases}$$

For $\theta_1^2 + \theta_2^2 \le d^2$, $|f_{,\alpha}| = |\theta_\beta f_{,\alpha\beta}(\overline{\theta}_1,\overline{\theta}_2)| = \mathcal{O}(\frac{d}{R}) = \mathcal{O}(\theta_0\theta)$ from (5.3). Treating the higher derivatives in a similar manner, it is seen that $f_{,\alpha} = G_0$. Using (5.7) and $\vec{r} = (\theta_1,\theta_2,f)$, it follows that

$$\vec{g}_1 = (1,0,G_0)$$

$$\vec{g}_2 = (0,1,G_0)$$

$$\vec{g}_1 \times \vec{g}_2 = (G_0,G_0,1)$$

$$|\vec{g}_1 \times \vec{g}_2| = 1 + G_0$$

$$\vec{g}_3 = (G_0,G_0,1+G_0)$$

$$a^i_j = \delta_{ij} + G_0$$

$$(A^i_j) = (a^i_j)^{-1} = (\delta_{ij} + G_0)$$

$$g_{\alpha\beta} = \delta_{\alpha\beta} + G_0 \text{ (remember } g_{\alpha 3} = 0\ ,\ g_{33} = 1)$$

$$(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1} = (\delta_{\alpha\beta} + G_0)$$

and therefore

$$(5.8)\quad a^i_j - \delta^i_j\ ,\ A^i_j - \delta^i_j\ ,\ g_{\alpha\beta} - \delta_{\alpha\beta}\ , \text{ and } g^{\alpha\beta} - \delta_{\alpha\beta} \text{ are all } G_0\ .$$

Also

$$b_{\alpha\beta} = \vec{g}_3 \cdot \vec{g}_{\alpha,\beta} = G_1$$

$$b^{\alpha}_{\beta} = g^{\alpha\gamma} b_{\gamma\beta} = G_1$$

$$g_{\alpha\beta,\gamma} = G_1$$

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma w}(g_{\beta\gamma,\alpha} + g_{\alpha\gamma,\beta} - g_{\alpha\beta,\gamma}) = G_1 \ .$$

Consequently

(5.9) $\quad b_{\alpha\beta}$; b^{α}_{β} ; $g_{\alpha\beta,\gamma}$; and Γ^{i}_{jk} are all G_1 .

From these

(5.10) $\quad C = 1 + G_0$, $B^{\alpha}_{\beta} = \delta_{\alpha\beta} + G_0$ (remember $B^{\alpha}_{3} = B^{3}_{\alpha} = 0$, $B^{3}_{3} = 1$).

If A denotes any set of functions of $\theta_1, \theta_2, \theta_3$, then $H_k(A)$ is used to denote any sum of terms of the sort $G_{k-\ell} \dfrac{\partial^{\ell} a}{\partial\theta_1^{\ell_1} \partial\theta_2^{\ell_2}}$ ($\ell = 0,1,2,\ldots,k$) where a is in A. Also $J_k(A)$ is used to denote any sum of terms of the sort $G_{k-\ell} \dfrac{\partial^{\ell} a}{\partial\theta_1^{\ell_1} \partial\theta_2^{\ell_2} \partial\theta_3^{\ell_3}}$ ($\ell = 0,1,2,\ldots,k$). Then

$$(5.11) \quad \begin{cases} \dfrac{\partial}{\partial\theta_\alpha} H_k(A) = H_{k+1}(A) \ , \ \dfrac{\partial}{\partial\theta_i} J_k(A) = J_{k+1}(A) \ , \ \dfrac{\partial}{\partial\theta_3} H_k(A) = J_{k+1}(A) \\ G_0 H_k(A) = H_k(A) \ , \ G_0 J_k(A) = J_k(A) \end{cases}$$

If A denotes any set of functions of $\theta_1, \theta_2, \theta_3$, let $\dot{A}$ denote the set of all functions $a_{,\alpha}$ where a is in A, and let A' denote the set of all functions $a_{,i}$ where a is in A. Similarly $\ddot{A}$ denotes the set of functions $a_{,\alpha\beta}$; $\dot{A}'$ denotes the set of functions $a_{,\alpha i}$; and A'' denotes the set of functions $a_{,ij}$ for a in A, etc. The notation $\dot{A}^{(k)}$ will denote A with k dots, and $A'^{(k)}$ will denote A with k primes. If A denotes a single function, A will also be used to denote the set consisting of the one function. Then $\dot{A}$, A', etc. have the above meaning. The notation $A_{,3}$ denotes the set of all functions $a_{,3}$ and $\dot{A}_{,3}$ denotes the set of all functions $a_{,\alpha 3}$, etc., where a is in A.

Let A,B, and C be sets of functions of $\theta_1, \theta_2, \theta_3$.

AB is used to denote the set of all functions ab where a is in A and b is in B. Similarly the product of more than two sets is defined.

Notation such as

$$A = BB' + BH_k(C) + J_k(B)H_\ell(C)$$

means each function of A is a linear combination of functions from the set BB' (with constant coefficients depending at most on n and W) plus a linear combination of functions from B (with coefficients of type $H_k(C)$) plus a linear combination of functions of type $J_k(B)H_\ell(C)$ (with constant coefficients depending at most on n and W).

Similarly if a is a function, the notation

$$a = BB' + BH_k(C) + J_k(B)H_\ell(C)$$

means a is a linear combination of the same type mentioned above.

Now let U,L,M,E,T,P,Q,R,S,V,F, and $\overline{Q}$ be the sets of all U^i, L_{ij}, M_{ij}, E^i_j, T^i_j, $P^i{}_j$, $Q^i{}_j$, R_{ij}, S_{ij}, V_i, F^i, and $\overline{Q}^i$ respectively.

From (1.7),

$$U^i|_j = (U^i{}_{,k} + \Gamma^i_{\ell k}U^\ell)B^\ell_j = U^i{}_{,j} + G_1U + G_0U' = U^i{}_{,j} + J_1(U)$$

using (5.9), (5.10), and (5.11). Also

$$U_i|^j = g_{ik}g^{j\ell}U^k|_\ell = U^i{}_{,j} + J_1(U)$$

using (5.8), (5.11), and $g_{\alpha 3} = g^{\alpha 3} = 0$, $g_{33} = g^{33} = 1$.

With these and (5.5) one has

$$(5.12)\quad \begin{cases} R = J_1(U) + U'U' + U'J_1(U) + J_1^2(U) \\ T = M + R + Z \\ P = U'T + TJ_1(U) \\ S = R + Z + P \\ Q = M + S = T + P \end{cases}$$

From (5.4) and (5.3)

$$(5.13)\qquad \begin{cases} H_k(F) = \mathcal{O}\left(\dfrac{h\varepsilon}{d^{k+2}}\right) \\ J_k(F) = \mathcal{O}\left(\dfrac{\varepsilon}{d^{k+1}}\right) \\ J_k(\overline{Q}) = \mathcal{O}\left(\dfrac{h\varepsilon}{d^{k+1}}\right) . \end{cases}$$

Frequent use will be made of the following functions. Let

$$(5.14)\qquad \begin{cases} \zeta_1 = \zeta_2 = \zeta_3 = \left(1 - \dfrac{\theta_1^2 + \theta_2^2}{d^2}\right)^2 & \text{for } \sqrt{\theta_1^2 + \theta_2^2} \le d \\ \zeta_k = \left[1 - \dfrac{4^{k-3}(\theta_1^2 + \theta_2^2)}{d^2}\right]^2 & \text{for } \sqrt{\theta_1^2 + \theta_2^2} \le \dfrac{d}{2^{k-3}} \\ & \text{where } k = 4,5,6,\ldots \\ \zeta_k = 0 \text{ otherwise } (k = 1,2,3,\ldots) \end{cases}$$

Then ζ_k and its first derivatives are continuous everywhere. ζ will be used to denote any one ζ_k.

Let A be any finite set of functions of $\theta_1, \theta_2, \theta_3$. Then $|A|^2$ will denote the sum of the squares of all functions in A with $|A| \ge 0$, and $\|A\|_k^2$ will denote the integral of $|A|^2$ over the region where $\zeta_k \ne 0$ and $|\theta_3| \le h$ ($\|A\|_k \ge 0$). The subscript on $\|A\|_k$ is omitted if the range of integration is clear from the context or if all values of k are permitted.

Easily established results for the functions ζ_k are

$$(5.15)\quad \begin{cases} |\zeta| \leq 1 \\ |\dot{\zeta}| = \mathcal{O}\left(\frac{1}{d}\sqrt{\zeta}\right) = \mathcal{O}\left(\frac{1}{d}\right) \\ |\ddot{\zeta}| = \mathcal{O}\left(\frac{1}{d^2}\right) \\ \left.\begin{array}{l} |\dot{\zeta}_{k+1}| = \mathcal{O}\left(\frac{1}{d}|\zeta_k|\right) \\ |\ddot{\zeta}_{k+1}| = \mathcal{O}\left(\frac{1}{d^2}|\zeta_k|\right) \end{array}\right\} \text{ for } k = 3,4,5,\ldots \end{cases}$$

If a, b, and c are any positive functions, the notation

$$a = \mathcal{O}(b) + \sigma(c)$$

will mean that for each $k > 0$ there is a constant $\bar{k} > 0$ depending only on k, W, and n such that

$$a \leq \bar{k}b + kc \ .$$

Frequent use will be made of the relations

$$|AB| = \mathcal{O}(|A|) + \sigma(|B|)$$

$$\|AB\| = \mathcal{O}(\|A\|) + \sigma(\|B\|)$$

where A and B are finite sets of functions.

Let $A_k = \max\|\zeta_k a\|_k$ where the maximum is taken over all functions a in a_k. Let $B_k = \max\|\zeta_k b\|_k$ where the maximum is taken over all functions b in b_k. Here $k = 1,2,3,\ldots$.

6. An estimate for the L_2 norms of V_i and its derivatives.

In this section the following is established for $k \geq 0$.

$$6.1 \quad \begin{cases} \left\|\zeta \dfrac{\partial^k V_i}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}}\right\| = \mathcal{O}\left[\dfrac{\varepsilon\sqrt{h}}{d^k} + \|\zeta\dot{M}^{(k+1)}\|\right. \\ \qquad\qquad \left. + \|\zeta S'^{(k+1)}\| + \|\zeta J_{k+1}(Q)\|\right] \\ \text{if } k_3 = 0\text{, or if } k_3 \text{ is even and } i \neq 3\text{, or if } k_3 \text{ is odd} \\ \text{and } i = 3. \\ \|\zeta V'^{(k)}\| = \mathcal{O}\left[\dfrac{\varepsilon}{\sqrt{h}\,d^{k-1}} + \|\zeta\dot{M}^{(k+1)}\| + \displaystyle\sum_{\alpha,\beta}\sum_{k_1,k_2}\left\|\zeta\dfrac{\partial^{k+1}M_{\alpha\beta}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3}\right\|\right. \\ \qquad\qquad \left. + \|\zeta S'^{(k+1)}\| + \|\zeta J_{k+1}(Q)\|\right] \end{cases}$$

All integrations are over the region where $\zeta \neq 0$ and $|\theta_3| \leq h$.

To establish these, use is made of the expansions in terms of Legendre polynomials of the various functions.

If $g(\theta_1,\theta_2,\theta_3)$ is defined for $|\theta_3| \leq h$, let

$$g_{(k)} = \frac{2k+1}{2}\int_{-1}^{1} g(\theta_1,\theta_2,\theta_3)\ P_k(\frac{\theta_3}{h})d(\frac{\theta_3}{h}) \text{ where}$$

$$P_k(x) = \frac{1}{2^k k!}\ \frac{d^k}{dx^k}\ (x^2-1)^k \text{ is the kth Legendre polynomial.}$$

Then from (2.8)

$$\int_{-h}^{h} (Q^\alpha{}_j|^j + F^\alpha)CP_k(\frac{\theta_3}{h})\ d\theta_3 = \left[(Q^\alpha{}_3 - \bar{Q}^\alpha)CP_k(\frac{\theta_3}{h})\right]_{\theta_3=-h}^{\theta_3=h}$$

for $k = 0,1,2,\ldots,n$.

Since $P_k(+1) = +1$ and $P_k(-1) = (-1)^k$, this becomes

$$\frac{2h}{2k+1}\left[(Q^\alpha{}_j|^j + F^\alpha)C\right]_{(k)} = \left[(Q^\alpha{}_3-\bar{Q}^\alpha)C\right]_{\theta_3=h} + (-1)^{k+1}\left[(Q^\alpha{}_3-\bar{Q}^\alpha)C\right]_{\theta_3=-h}$$

for $k = 0,1,\ldots,n$.

Since the right hand side does not change as k increases by two's, one obtains

$$\left[(Q^\alpha{}_j|^j + F^\alpha)C\right]_{(k)} = \frac{2k+1}{2n+1}\left[(Q^\alpha{}_j|^j + F^\alpha)C\right]_{(n)} \text{ if } k+n \text{ is even } (k=0,1,\ldots,n),$$

$$\left[(Q^\alpha{}_j|^j + F^\alpha)C\right]_{(k)} = \frac{2k+1}{2n-1}\left[(Q^\alpha{}_j|^j + F^\alpha)C\right]_{(n-1)} \text{ if } k+n \text{ is odd } (k=0,1,\ldots,n).$$

Hence

$$\sum_{k=0}^{n}\left[(Q^{\alpha}{}_{j}|^{j}+F^{\alpha})C\right]_{(k)}P_k(\frac{\theta_3}{h}) = \frac{1}{2n+1}\left[(Q^{\alpha}{}_{j}|^{j}+F^{\alpha})C\right]_{(n)}\sum_{\substack{k=0\\k+n\ \text{even}}}^{n}(2k+1)P_k(\frac{\theta_3}{h})$$

$$+\frac{1}{2n-1}\left[(Q^{\alpha}{}_{j}|^{j}+F^{\alpha})C\right]_{(n-1)}\sum_{\substack{k=0\\k+n\ \text{odd}}}^{n}(2k+1)P_k(\frac{\theta_3}{h}) \ .$$

From the identity

$$(2k+1)P_k(x) = P'_{k+1}(x) - P'_{k-1}(x) \ , \quad k \geq 1$$

one has

$$\sum_{\substack{k=0\\k+n\ \text{even}}}^{n}(2k+1)P_k(\frac{\theta_3}{h}) = P'_{n+1}(\frac{\theta_3}{h}) \ ,$$

$$\sum_{\substack{k=0\\k+n\ \text{odd}}}^{n}(2k+1)P_k(\frac{\theta_3}{h}) = P'_{n}(\frac{\theta_3}{h}) \ .$$

Hence

$$\sum_{k=0}^{n}\left[(Q^{\alpha}{}_{j}|^{j}+F^{\alpha})C\right]_{(k)}P_k(\frac{\theta_3}{h}) \tag{6.2}$$

$$= \sum_{k=n-1}^{n}\frac{1}{2k+1}\left[(Q^{\alpha}{}_{j}|^{j}+F^{\alpha})C\right]_{(k)}P'_{k+1}(\frac{\theta_3}{h}) \ .$$

$$V_\alpha = \sum_{k=0}^{n} V_{\alpha(k)} P_k\left(\frac{\theta_3}{h}\right) = \sum_{k=0}^{n} (Q^\alpha{}_{j,j} - S_{\alpha j,j})_{(k)} P_k\left(\frac{\theta_3}{h}\right) \text{(from the definitions of } S_{ij} \text{ and } V_i)$$

$$= \sum_{k=0}^{n} [CQ^\alpha{}_j|^j + J_1(Q) + S']_{(k)} P_k\left(\frac{\theta_3}{h}\right) \text{ from (6.4)}$$

$$= \sum_{k=0}^{n} [-CF^\alpha + J_1(Q) + S']_{(k)} P_k\left(\frac{\theta_3}{h}\right)$$

$$+ \sum_{k=n-1}^{n} \frac{1}{2k+1} P'_{k+1}\left(\frac{\theta_3}{h}\right)[(Q^\alpha{}_j|^j + F^\alpha)C]_{(k)} \text{ from (6.2)}$$

$$= \sum_{k=0}^{n} [-F^\alpha + H_0(F) + J_1(Q) + S']_{(k)} P_k\left(\frac{\theta_3}{h}\right)$$

$$+ \sum_{k=n-1}^{n} \frac{1}{2k+1} P'_{k+1}\left(\frac{\theta_3}{h}\right)[V_\alpha + S' + J_1(Q) + F^\alpha + H_0(F)]_{(k)}$$

from (6.4)

For $k = n-1, n$

$$V_{\alpha(k)} = [M_{\alpha\beta,\beta} + \mu(U^\alpha{}_{,33} + U^3{}_{,3\alpha})]_{(k)} \text{(from the definitions of } M_{\alpha 3}),$$

$$= (M_{\alpha\beta,\beta} + \mu U^3{}_{,3\alpha})_{(k)}$$

since $U^\alpha{}_{,33}$ is an $(n-2)$nd degree polynomial in θ_3 if $n \geq 2$ and is zero if $n = 1$.

Similarly

$$\sum_{k=0}^{n+1} \left[(Q^3{}_j|^j + F^{\alpha})C\right]_{(k)} P_k\left(\frac{\theta_3}{h}\right) =$$

(6.3

$$= \sum_{k=n}^{n+1} \frac{1}{2k+1} \left[(Q^3{}_j|^j + F^3)C\right]_{(k)} P'_{k+1}\left(\frac{\theta_3}{h}\right) .$$

From (1.8), (5.9), and (5.10)

$$(6.4)\begin{cases} Q^i{}_j|^j = (Q^i{}_{j,k} + \Gamma^i_{\ell k} Q^{\ell}{}_j - \Gamma^{\ell}_{jk} Q^i{}_{\ell}) B^{kj} = Q^i{}_{j,j} + G_1 Q + G_0 \dot{Q} \\ \qquad = Q^i{}_{j,j} + H_1(Q) , \\ C(Q^i{}_j|^j - Q^i{}_{j,j}) = (1 + G_0) H_1(Q) = H_1(Q) \\ CQ^i{}_j|^j = (1 + G_0) Q^i{}_{j,j} + H_1(Q) = Q^i{}_{j,j} + J_1(Q) \\ \qquad = V_i + S' + J_1(Q) \end{cases}$$

Considering V_i as a function of displacements, it is seen that V_α is an nth degree polynomial in θ_3 and V_3 is an (n+1)st degree polynomial. Hence

Hence

$$V_\alpha = \sum_{k=0}^{n} [-F^\alpha + J_0(F) + J_1(Q) + S']_{(k)} P_k(\frac{\theta_3}{h})$$

(6.5)

$$+ \sum_{k=n-1}^{n} \frac{1}{2k+1} P'_{k+1}(\frac{\theta_3}{h}) [M_{\alpha\beta,\beta} + \mu U^3{}_{,3\alpha} + S'$$

$$+ J_1(Q) + F^\alpha + J_0(F)]_{(k)}$$

Similarly

$$V_3 = \sum_{k=0}^{n+1} [-F^3 + H_0(F) + J_1(Q) + S']_{(k)} P_k(\frac{\theta_3}{h})$$

$$+ \sum_{k=n}^{n+1} \frac{1}{2k+1} P'_{k+1}(\frac{\theta_3}{h}) [V_3 + S' + J_1(Q) + F^3 + H_0(F)]_{(k)} .$$

For $k = n, n+1$

$$V_{3(k)} = [M_{3\beta,\beta} + \lambda U^\beta{}_{,\beta 3} + (\lambda + 2\mu) U^3{}_{,33}]_{(k)} = (M_{3\beta,\beta})_{(k)}$$

since $U^\beta{}_{,\beta 3}$ and $U^3{}_{,33}$ are both $(n-1)$st degree polynomials in θ_3. Hence

$$V_3 = \sum_{k=0}^{n+1} [-F^3 + J_0(F) + J_1(Q) + S']_{(k)} P_k(\frac{\theta_3}{h})$$

(6.6)

$$+ \sum_{k=n}^{n+1} \frac{1}{2k+1} P'_{k+1}(\frac{\theta_3}{h}) [M_{3\beta,\beta} + S' + J_1(Q) + F^3 + J_0(F)]_{(k)} .$$

Lemma (6.7): Let A and B be finite sets of functions (the number depending only on n) such that

$$a = \sum_i \sum_{k=0}^{m} b_{i(k)} p_k\left(\frac{\theta_3}{h}\right)$$

for every function a in A where each b_i is in B, p_k is a kth degree polynomial whose coefficients depend only on n, and m depends only on n. Then

$$\|A\| = \mathcal{O}(\|B\|)$$

$$\left\|\frac{\partial^k A}{\partial\theta^{k_1}\partial\theta^{k_2}\partial\theta^{k_3}}\right\| = \mathcal{O}\left(\left\|\frac{\partial^k B}{\partial\theta^{k_1}\partial\theta^{k_2}\partial\theta^{k_3}}\right\|\right)$$

for all k for which the derivatives are continuous.

From Bessel's inequality

$$|a|^2 = \mathcal{O}\left(\sum_i \sum_{k=0}^{m} b^2_{i(k)}\right) = \mathcal{O}\left[\sum_i \int_{-1}^{1} b_i^2 d\left(\frac{\theta_3}{h}\right)\right]$$

so that

$$\int_{-h}^{h} |a|^2 d\theta_3 = \mathcal{O}\left(\sum_i \int_{-h}^{h} b_i^2 d\theta_3\right) \quad \text{, and}$$

$$\|a\|^2 = \mathcal{O}(\|B\|^2) \text{ .}$$

Hence $\|A\| = \mathcal{O}(\|B\|)$.

Since $a_{,\alpha} = \sum_i \sum_{k=0}^{m} (b_{i,\alpha})_{(k)} p_k(\frac{\theta_3}{h})$, then $\|A_{,\alpha}\| = \mathcal{O}(\|B_{,\alpha}\|)$ from the previous argument.

Next observe that for $k \geq 1$

$$b_{i(k)} = \frac{2k+1}{2}\int_{-1}^{1} b_i P_k(\frac{\theta_3}{h})d(\frac{\theta_3}{h}) = \frac{1}{2}\int_{-1}^{1} b_i[P'_{k+1}(\frac{\theta_3}{h}) - P'_{k-1}(\frac{\theta_3}{h})]d(\frac{\theta_3}{h})$$

$$= \frac{1}{2}\int_{-h}^{h} b_i \frac{d}{d\theta_3}[P_{k+1}(\frac{\theta_3}{h}) - P_{k-1}(\frac{\theta_3}{h})]d\theta_3$$

$$= -\frac{1}{2}\int_{-h}^{h} b_{i,3}[P_{k+1}(\frac{\theta_3}{h}) - P_{k-1}(\frac{\theta_3}{h})]d\theta_3$$

$$= \frac{h}{2k-1}(b_{i,3})_{(k-1)} - \frac{h}{2k+3}(b_{i,3})_{(k+1)}$$

since $(2k+1)P_k(x) = P'_{k+1}(x) - P'_{k-1}(x)$ and $P_{k+1}(\pm 1) - P_{k-1}(\pm 1) = 0$.

But

$$a_{,3} = \frac{1}{h}\sum_i \sum_{k=1}^{m} b_{i(k)} p'_k(\frac{\theta_3}{h})$$

$$= \sum_i \sum_{k=1}^{m} [\frac{1}{2k-1}(b_{i,3})_{(k-1)} - \frac{1}{2k+3}(b_{i,3})_{(k+1)}]p'_k(\frac{\theta_3}{h}) .$$

Then $\|A_{,3}\| = \mathcal{O}(\|B_{,3}\|$ by the argument used in the first part of the lemma. The differentiations can be repeated to obtain the lemma.

From (6.5), (6.6), and the lemma

$$
(6.7)\begin{cases}
\left\| \zeta \dfrac{\partial^k V_\alpha}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}} \right\| = \mathcal{O}\left[\left\| \zeta \dfrac{\partial^k M_{\alpha\beta,\beta}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}} \right\| + \left\| \zeta \dfrac{\partial^k U^3{}_{,3\alpha}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}} \right\| \right. \\
\qquad \left. + \left\| \zeta \dfrac{\partial^k F^\alpha}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}} \right\| + \| \zeta S^{\prime(k+1)} \| + \| \zeta J_{k+1}(Q) \| + \| J_k(F) \| \right] \\
\left\| \zeta \dfrac{\partial^k V_3}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}} \right\| = \mathcal{O}\left[\left\| \zeta \dfrac{\partial^k M_{3\beta,\beta}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}} \right\| + \left\| \zeta \dfrac{\partial^k F^3}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}} \right\| \right. \\
\qquad \left. + \| \zeta S^{\prime(k+1)} \| + \| \zeta J_{k+1}(Q) \| + \| \zeta J_k(F) \| \right]
\end{cases}
$$

If $k_3 = 0$

$$
\left\| \zeta \frac{\partial^k U^3{}_{,3\alpha}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}} \right\| = \mathcal{O}[\| \zeta \dot{M}^{(k+1)} \|] \text{ from } (4.1)
$$

and (6.1) follows from (6.7) for $k_3 = 0$ using (5.4) and (5.13).

If $k_3 = 1$, Theorem (4.1) gives

$$
\left\| \zeta \frac{\partial^k U^3{}_{,3\alpha}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3} \right\| = \mathcal{O}[\| \zeta \dot{M}^{(k+1)} \| + \sum_{\beta,\gamma} \| \zeta \dot{M}^{(k)}_{\beta\gamma,3} \| + \| \zeta \dot{V}^{(k)} \|]
$$

$$
\left\| \zeta \frac{\partial^k M_{3\beta,\beta}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3} \right\| = \mathcal{O}[\| \zeta \dot{M}^{(k+1)} \| + \| \zeta \dot{V}^{(k)} \|] .
$$

Thus (6.1) follows for $k_3 = 1$ from (6.7) using (6.1) for the case $k_3 = 0$.

For $k_3 = 2$, Theorem (4.1) gives

$$\left\|\zeta\frac{\partial^k M_{\alpha\beta,\beta}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^2}\right\| = \mathcal{O}[\|\zeta\dot{M}^{(k+1)}\| + \|\zeta\dot{V}^{(k)}\| + \|\zeta\dot{V}_{3,3}^{(k-1)}\|]$$

$$\left\|\zeta\frac{\partial^k U^3_{,3\alpha}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^2}\right\| = \mathcal{O}[\|\zeta\dot{M}^{(k+1)}\| + \|\zeta\dot{V}^{(k)}\| + \|\zeta\dot{V}_{3,3}^{(k-1)}\|]$$

$$\left\|\zeta\frac{\partial^k M_{3\beta,\beta}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^2}\right\| = \mathcal{O}[\|\zeta\dot{M}^{(k+1)}\| + \sum_{\alpha,\gamma}\|\zeta\dot{M}_{\alpha\gamma,3}^{(k)}\| + \|\zeta\dot{V}^{(k)}\| + \|\zeta\dot{V}_{,3}^{(k-1)}\|] .$$

Thus (6.1) follows for $k_3 = 2$ from (6.7) using (6.1) for the cases $k_3 = 0,1$.

Proceeding in this manner (6.1) is proved for all k_3.

7. L_2 estimates for functions in the set $\dot{M}^{(k)}$.

For $k \geq 3$ the following is established:

$$\begin{aligned}
\|\zeta_k\dot{M}^{(k-1)}\| = \mathcal{O}\Big\{ & \frac{\sqrt{h}\,\varepsilon}{d^{k-2}} + \|\dot{\zeta}_k\dot{M}^{(k-2)}\| + \frac{1}{d^2}\|\dot{M}^{(k-3)}\|_k + \|\zeta_k R'^{(k-1)}\| \\
& + \|\zeta_k Z'^{(k-1)}\| \\
& + \frac{d}{h}[\|\zeta_k H_{k-1}(Q)\| + \|\zeta_k\dot{P}^{(k-1)}\| + \|\zeta_k H_{k-1}(P)\| + \|\zeta_k H_{k-1}(T)\|] \\
& + \sqrt{\left|\iiint \zeta_k^2 H_k(Q)\dot{U}^{(k-1)}\right|} + \sqrt{\left|\iiint \zeta_k\dot{\zeta}_k H_k(Q)\dot{U}^{(k-2)}\right|}\Big\} \\
& + \sigma\Big\{\frac{1}{d}A_{k-1} + A_k + \frac{h}{d^3}B_{k-2} + \frac{h}{d^2}B_{k-1} + \frac{h}{d}B_k + \frac{h}{d}\|\dot{\zeta}_k U'^{(k-1)}\|\Big\}
\end{aligned} \tag{7.1}$$

where all integrations are over the region where $\zeta_k \neq 0$ and $|\theta_3| \leq h$.

Using the divergence theorem,

$$\iiint \zeta_k^2 CQ^i{}_j U^i{}_{,j} = -\iiint [\zeta_k^2(CQ^i{}_{j,j} + C_{,j}Q^i{}_j)U^i + (\zeta_k^2)_{,\alpha}CQ^i{}_\alpha U^i]$$

$$+ \iint \zeta_k^2 (CQ^i{}_3 U^i)_{\theta_3=-h}^{\theta_3=h} \, d\theta_1 d\theta_2 \ .$$

From (2.8)

$$\iint \zeta_k^2 (CQ^i{}_3 U^i)_{\theta_3=-h}^{\theta_3=h} \, d\theta_1 d\theta_2$$

$$= \iint \zeta_k^2 [C(Q^i{}_3 - \overline{Q}^i)U^i]_{\theta_3=-h}^{\theta_3=h} \, d\theta_1 d\theta_2 + \iiint \zeta_k^2 (C\overline{Q}^i U^i)_{,3}$$

$$= \iiint \zeta_k^2 [(Q^i{}_j|^j + F^i)CU^i + (C\overline{Q}^i U^i)_{,3}] \ .$$

Thus

$$\iiint \zeta_k^2 CQ^i{}_j U^i{}_{,j}$$

$$= \iiint \Big\{ \zeta_k^2 [C(Q^i{}_j|^j + F^i - Q^i{}_{j,j}) - C_{,j}Q^i{}_j]U^i + \zeta_k^2 (C\overline{Q}^i U^i)_{,3}$$

$$- (\zeta_k)^2{}_{,\alpha} CQ^i{}_\alpha U^i \Big\}$$

$$= \iiint \Big\{ \zeta_k^2 [F^i U^i + (\overline{Q}^i U^i)_{,3}] + \zeta_k^2 [H_1(Q) + H_0(F) + J_1(\overline{Q})]U$$

$$+ \zeta_k^2 J_0(\overline{Q})U' - (\zeta_k^2)_{,\alpha} CQ^i{}_\alpha U^i \Big\} \quad \text{from (6.4) and (5.10).}$$

By exactly the same argument

$$\left|\iiint \zeta_k^2 (CQ^i{}_j)_{,\alpha_1\alpha_2\ldots\alpha_{k-1}} U^i{}_{,j\alpha_1\alpha_2\ldots\alpha_{k-1}}\right|$$

$$= \left|\iiint \Big\{ \zeta_k^2 \big[F^i{}_{,\alpha_1\alpha_2\ldots\alpha_{k-1}} U^i{}_{,\alpha_1\alpha_2\ldots\alpha_{k-1}}\right.$$

$$+ (\overline{Q}^i{}_{,\alpha_1\alpha_2\ldots\alpha_{k-1}} U^i{}_{,\alpha_1\alpha_2\ldots\alpha_{k-1}})_{,3}\big]$$

$$+ \zeta_k^2 [H_k(Q) + H_{k-1}(F) + J_k(\overline{Q})] \dot{U}^{(k-1)} + \zeta_k^2 J_{k-1}(\overline{Q}) U'^{(k)}$$

$$\left. - (\zeta_k^2)_{,\alpha} (CQ^i{}_\alpha)_{,\alpha_1\alpha_2\ldots\alpha_{k-1}} U^i{}_{,\alpha_1\alpha_2\ldots\alpha_{k-1}} \Big\} \right|$$

$$= \mathcal{C} \Big\{ \frac{h\varepsilon^2}{d^{2k-4}} + \Big| \iiint \big[\zeta_k^2 H_k(Q) \dot{U}^{(k-1)}$$

$$- (\zeta_k^2)_{,\alpha} (CQ^i{}_\alpha)_{,\alpha_1\alpha_2\ldots\alpha_{k-1}} U^i{}_{,\alpha_1\alpha_2\ldots\alpha_{k-1}} \big] \Big| \Big\}$$

$$+ \sigma\left(\frac{1}{d^2} A_{k-1}^2 + \frac{h^2}{d^4} B_{k-1}^2 + \frac{h^2}{d^2} B_k^2 + A_k^2 \right) \text{ using (5.13) and (5.4).}$$

Next

$$\left|\iiint (\zeta_k^2)_{,\alpha}(CQ^i{}_\alpha)_{,\alpha_1\alpha_2\ldots\alpha_{k-1}} U^i{}_{,\alpha_1\alpha_2\ldots\alpha_{k-1}}\right|$$

$$= \left|\iiint (\zeta_k^2)_{,\alpha}(CQ^i{}_\alpha)_{,\beta\alpha_1\alpha_2\ldots\alpha_{k-2}} U^i{}_{,\beta\alpha_1\alpha_2\ldots\alpha_{k-2}}\right|$$

$$= \left|\iiint (\zeta_k^2)_{,\alpha}(CQ^i{}_\alpha)_{,\beta\alpha_1\alpha_2\ldots\alpha_{k-2}} (2L_{i\beta} - U^\beta{}_{,i})_{,\alpha_1\alpha_2\ldots\alpha_{k-2}}\right|$$

$$= \left|\iiint \Big\{ \zeta_k\dot{\zeta}_k[\dot{Q}^{(k-1)} + H_{k-1}(Q)]\dot{L}^{(k-2)}\right.$$

$$\left. - (\zeta_k^2)_{,\alpha}(CQ^i{}_\alpha)_{,\beta\alpha_1\alpha_2\ldots\alpha_{k-2}} U^\beta{}_{,i\alpha_1\alpha_2\ldots\alpha_{k-2}} \Big\}\right|$$

$$= \mathcal{O}\Big\{\|\zeta_k H_{k-1}(Q)\|^2 + \|\dot{\zeta}_k \dot{M}^{(k-2)}\|^2$$

$$+ \left|\iiint (\zeta_k^2)_{,\alpha}(CQ^i{}_\alpha)_{,\beta\alpha_1\alpha_2\ldots\alpha_{k-2}} U^\beta{}_{,i\alpha_1\alpha_2\ldots\alpha_{k-2}}\right|\Big\} + \sigma(A_k^2)$$

using the fact that L_{ij} is a linear combination of the $M_{k\ell}$.

Since $Q^i{}_\alpha = T^i_\alpha + P^i{}_\alpha = Q_\alpha{}^i + P^i{}_\alpha - P_\alpha{}^i = Q^\alpha{}_i + H_0(Q) + P + H_0(P)$ from (5.8), then $CQ^i{}_\alpha = CQ^\alpha{}_i + H_0(Q) + P + H_0(P)$ using (5.10).

Thus

$$\left|\iiint (\zeta_k^2)_{,\alpha}(CQ^i{}_\alpha)_{,\beta\alpha_1\alpha_2\ldots\alpha_{k-2}}U^\beta{}_{,i\alpha_1\alpha_2\ldots\alpha_{k-2}}\right|$$

$$= \left|\iiint \Big\{\zeta_k\dot\zeta_k[H_{k-1}(Q) + \dot P^{(k-1)} + H_{k-1}(P)]U'^{(k-1)}\right.$$

$$\left. + (\zeta_k^2)_{,\alpha}(CQ^\alpha{}_i)_{,\beta\alpha_1\alpha_2\ldots\alpha_{k-2}}U^\beta{}_{,i\alpha_1\alpha_2\ldots\alpha_{k-2}}\Big\}\right|$$

$$= \mathcal{C}\Big\{\frac{d^2}{h^2}[\|\zeta_k H_{k-1}(Q)\|^2 + \|\zeta_k\dot P^{(k-1)}\|^2 + \|\zeta_k H_{k-1}(P)\|^2]$$

$$+ \left|\iiint (\zeta_k^2)_{,\alpha}(CQ^\alpha{}_i)_{,\beta\alpha_1\alpha_2\ldots\alpha_{k-2}}U^\beta{}_{,i\alpha_1\alpha_2\ldots\alpha_{k-2}}\right|\Big\}$$

$$+ \sigma[\frac{h^2}{d^2}\|\dot\zeta_k U'^{(k-1)}\|^2]$$

Using (2.8) and (6.4) again

$$\iiint (\zeta_k^2)_{,\alpha}(CQ^\alpha{}_i)_{,\beta}U^\beta{}_{,i}$$

$$= -\iiint [(\zeta_k^2)_{,\alpha}(CQ^\alpha{}_{i,i} + C_{,i}Q^\alpha{}_i)_{,\beta}U^\beta + (\zeta_k^2)_{,\alpha\gamma}(CQ^\alpha{}_\gamma)_{,\beta}U^\beta]$$

$$+ \iint (\zeta_k^2)_{,\alpha}[(CQ^\alpha{}_3)_{,\beta}U^\beta]_{\theta_3=-h}^{\theta_3=h}d\theta_1 d\theta_2$$

$$= \iiint \Big\{(\zeta_k^2)_{,\alpha}[C(Q^\alpha{}_j|^j + F^\alpha - Q^\alpha{}_{i,i}) - C_{,i}Q^\alpha{}_i]_{,\beta}U^\beta$$

$$+ (\zeta_k^2)_{,\alpha}[(C\overline{Q}^\alpha)_{,\beta}U^\beta]_{,3} - (\zeta_k^2)_{,\alpha\gamma}(CQ^\alpha{}_\gamma)_{,\beta}U^\beta\Big\}$$

$$= \iiint \Big\{(\zeta_k^2)_{,\alpha}[F^\alpha{}_{,\beta}U^\beta + (\overline{Q}^\alpha{}_{,\beta}U^\beta)_{,3}]$$

$$+ \zeta_k\dot\zeta_k[H_2(Q) + H_1(F) + J_2(\overline{Q})]U + \zeta_k\dot\zeta_k J_1(\overline{Q})U'$$

$$- (\zeta_k^2)_{,\alpha\gamma}(CQ^\alpha{}_\gamma)_{,\beta}U^\beta\Big\} .$$

By the same argument

$$\left| \iiint (\zeta_k^2)_{,\alpha} (CQ^\alpha{}_i)_{,\beta\alpha_1\alpha_2\cdots\alpha_{k-2}} U^\beta{}_{,i\alpha_1\alpha_2\cdots\alpha_{k-2}} \right|$$

$$= \left| \iiint \Big\{ (\zeta_k^2)_{,\alpha} [F^\alpha{}_{,\beta\alpha_1\alpha_2\cdots\alpha_{k-2}} U^\beta{}_{,\alpha_1\alpha_2\cdots\alpha_{k-2}}\right.$$

$$+ (\overline{Q}^\alpha{}_{,\beta\alpha_1\alpha_2\cdots\alpha_{k-2}} U^\beta{}_{,\alpha_1\alpha_2\cdots\alpha_{k-2}})_{,3}]$$

$$+ \zeta_k \dot{\zeta}_k [H_k(Q) + H_{k-1}(F) + J_k(\overline{Q})] \dot{U}^{(k-2)} + \zeta_k \dot{\zeta}_k J_{k-1}(\overline{Q}) U'^{(k-1)}$$

$$\left. - (\zeta_k^2)_{,\alpha\gamma} (CQ^\alpha{}_\gamma)_{,\beta\alpha_1\alpha_2\cdots\alpha_{k-2}} U^\beta{}_{,\alpha_1\alpha_2\cdots\alpha_{k-2}} \Big\} \right|$$

But

$$\left| \iiint (\zeta_k^2)_{,\alpha} [F^\alpha{}_{,\beta\alpha_1\alpha_2\cdots\alpha_{k-2}} + \bar{Q}^\alpha{}_{,\beta 3\alpha_1\alpha_2\cdots\alpha_{k-2}}] U^\beta{}_{,\alpha_1\alpha_2\cdots\alpha_{k-2}} \right|$$

$$= \left| \iiint (\zeta_k^2)_{,\alpha} (F^\alpha{}_{,\beta\gamma\alpha_1\alpha_2\cdots\alpha_{k-3}} + \right.$$

$$\left. + \bar{Q}^\alpha{}_{,\beta\gamma 3\alpha_1\alpha_2\cdots\alpha_{k-3}}) L_{\beta\gamma,\alpha_1\alpha_2\cdots\alpha_{k-3}} \right|$$

$$= \mathcal{O}\left(\frac{h\varepsilon^2}{d^{2k-4}}\right) + \sigma\left[\frac{1}{d^4} \|\zeta_k \dot{L}^{(k-3)}\|^2\right] \quad \text{using (5.4)}$$

$$= \mathcal{O}\left(\frac{h\varepsilon^2}{d^{2k-4}}\right) + \sigma\left[\frac{1}{d^4} \|\zeta_k \dot{M}^{(k-3)}\|^2\right] .$$

Thus

$$\left|\iiint (\zeta_k^2)_{,\alpha}(CQ^{\alpha}{}_{i})_{,\beta\alpha_1\alpha_2\ldots\alpha_{k-2}} U^{\beta}{}_{,i\alpha_1\alpha_2\ldots\alpha_{k-2}}\right|$$

$$= \mathcal{O}\left\{\frac{h\varepsilon^2}{d^{2k-4}} + \left|\iiint [\zeta_k\dot{\zeta}_k H_k(Q)\dot{U}^{(k-2)} - (\zeta_k^2)_{,\alpha\gamma}(CQ^{\alpha}{}_{\gamma})_{,\beta\alpha_1\alpha_2\ldots\alpha_{k-2}} U^{\beta}{}_{,\alpha_1\alpha_2\ldots\alpha_{k-2}}]\right|\right\}$$

$$+ \sigma\left(\frac{1}{d^4}\|\zeta_k\dot{M}^{(k-3)}\|^2 + \frac{h^2}{d^4}B_{k-1}^2 + \frac{h^2}{d^6}B_{k-2}^2\right)$$ using (5.4) and (5.13).

Next

$$\left|\iiint (\zeta_k^2)_{,\alpha\gamma}(CQ^{\alpha}{}_{\gamma})_{,\beta\alpha_1\alpha_2\ldots\alpha_{k-2}} U^{\beta}{}_{,\alpha_1\alpha_2\ldots\alpha_{k-2}}\right|$$

$$= \left|\iiint (\zeta_k^2)_{,\alpha\gamma}(CQ^{\alpha}{}_{\gamma})_{,\beta\omega\alpha_1\alpha_2\ldots\alpha_{k-3}} U^{\beta}{}_{,\omega\alpha_1\alpha_2\ldots\alpha_{k-3}}\right|$$

$$= \left|\iiint (\zeta_k^2)_{,\alpha\gamma}(CQ^{\alpha}{}_{\gamma})_{,\beta\omega\alpha_1\alpha_2\ldots\alpha_{k-3}} L_{\beta\omega,\alpha_1\alpha_2\ldots\alpha_{k-3}}\right|$$

$$= \mathcal{O}\left\{\left|\iiint (\dot{\zeta}_k^2 + \zeta_k\ddot{\zeta}_k)[\dot{Q}^{(k-1)} + H_{k-1}(Q)]\dot{L}^{(k-3)}\right|\right\}$$

$$= \mathcal{O}[\|\zeta_k H_{k-1}(Q)\|^2 + \frac{1}{d^4}\|\dot{M}^{(k-3)}\|_k^2] + \sigma(A_k^2) .$$

Collecting results

$$(7.2)\quad \left|\iiint \zeta_k^2 (CQ^i{}_j)_{,\alpha_1\alpha_2\ldots\alpha_{k-1}} U^i{}_{,j\alpha_1\alpha_2\ldots\alpha_{k-1}}\right|$$

$$= \mathcal{O}\left\{\frac{h\varepsilon^2}{d^{2k-4}} + \|\dot{\zeta}_k \dot{M}^{(k-2)}\|^2 + \frac{1}{d^4}\|\dot{M}^{(k-3)}\|_k^2\right.$$

$$+ \frac{d^2}{h^2}\left[\|\zeta_k H_{k-1}(Q)\|^2 + \|\zeta_k \dot{P}^{(k-1)}\|^2 + \|\zeta_k H_{k-1}(P)\|^2\right]$$

$$\left. + \left|\iiint \zeta_k^2 H_k(Q)\dot{U}^{(k-1)}\right| + \left|\iiint \zeta_k \dot{\zeta}_k H_k(Q)\dot{U}^{(k-2)}\right|\right\}$$

$$+ \sigma\left\{\frac{1}{d^2} A_{k-1}^2 + A_k^2 + \frac{h^2}{d^6} B_{k-2}^2\right.$$

$$\left. + \frac{h^2}{d^4} B_{k-1}^2 + \frac{h^2}{d^2} B_k^2 + \frac{h^2}{d^2}\|\dot{\zeta}_k U'^{(k-1)}\|^2\right\}.$$

Using $L_{ij} = \frac{1}{2}(U^i{}_{,j} + U^j{}_{,i})$ and the symmetry of T^{ij},

$$Q^i{}_j U^i{}_{,j} = (T^i_j + P^i{}_j)U^i{}_{,j} = (T^{ij} + G_0 T + P^i{}_j)U^i{}_{,j}$$

$$= T^{ij}L_{ij} + [H_0(T) + P]U'$$

$$= [T^i_j + H_0(T)]L_{ij} + [H_0(T) + P]U'$$

$$= T^i_j L_{ij} + [H_0(T) + P]U'$$

$$= (M_{ij} + R + Z)L_{ij} + [H_0(T) + P]U'$$

$$= M_{ij}L_{ij} + (R + Z)M + [H_0(T) + P]U'$$

But

$$M_{ij}L_{ij} = \lambda L_{kk}L_{ii} + 2\mu L_{ij}L_{ij} \geq 2\mu|L|^2$$

so that

$$|L|^2 = \mathcal{O}(M_{ij}L_{ij}) .$$

Since $|M| = \mathcal{O}(|L|)$, it follows that

$$|M|^2 = \mathcal{O}(M_{ij}L_{ij})$$

$$= \mathcal{O}\left\{Q^i{}_j U^i{}_{,j} + (R+Z)M + [H_0(T) + P]U'\right\}.$$

By the same argument

$$|\dot{M}^{(k-1)}|^2 = \mathcal{O}\Big\{Q^i{}_{j,\alpha_1\alpha_2\ldots\alpha_{k-1}} U^i{}_{,j\alpha_1\alpha_2\ldots\alpha_{k-1}}$$

$$+ [R'^{(k-1)} + Z'^{(k-1)}]\dot{M}^{(k-1)} + [H_{k-1}(T) + \dot{P}^{(k-1)}]U'^{(k)}\Big\}.$$

Since

$$(CQ^i{}_j)_{,\alpha_1\alpha_2\ldots\alpha_{k-1}} U^i{}_{,j\alpha_1\alpha_2\ldots\alpha_{k-1}}$$

$$= Q^i{}_{j,\alpha_1\alpha_2\ldots\alpha_{k-1}} U^i{}_{,j\alpha_1\alpha_2\ldots\alpha_{k-1}} + H_{k-1}(Q)U'^{(k)}$$

the above becomes

$$|\dot{M}^{(k-1)}|^2 = \mathcal{O}\Big\{(CQ^i{}_j)_{,\alpha_1\alpha_2\ldots\alpha_{k-1}} U^i{}_{,j\alpha_1\alpha_2\ldots\alpha_{k-1}}$$

$$+ [R'^{(k-1)} + Z'^{(k-1)}]\dot{M}^{(k-1)} + [H_{k-1}(T) + \dot{P}^{(k-1)} + H_{k-1}(Q)]U'^{(k)}\Big\}.$$

Using this with (7.2), one obtains (7.1).

8. L_2 estimates for functions in a_k and $b_k (k = 2,3)$.

Consider the range $\theta_1^2 + \theta_2^2 \leq d^2$, $|\theta_3| \leq h$, and let $\zeta = \zeta_1 = \zeta_2 = \zeta_3$.

Then

$$
(8.1)\quad \begin{cases}
|x_\alpha| = |\theta_\alpha| = \mathcal{O}(d) \text{ by choice of } \theta_\alpha \\
|x_3| = \mathcal{O}(|\theta_i| \max |\frac{\partial x_3}{\partial \theta_i}|) = \mathcal{O}[|\theta_i| \max |a_k^3(\delta_i^k - \theta_3 b_i^k)|] \\
\qquad = \mathcal{O}(|\theta_i||\delta_{3i} + G_0|) = \mathcal{O}(h + \theta_0 \theta d) = \mathcal{O}(h) \\
\qquad\qquad \text{from (1.5) and (5.8-9)} \\
|u_i| = \mathcal{O}(|x_j| \max |\frac{\partial u_i}{\partial x_j}|) = \mathcal{O}(\frac{d^2}{h} \varepsilon) \text{ from (5.4)} \\
|U^i| = |A_j^i u_j| = \mathcal{O}(\frac{d^2}{h} \varepsilon) \text{ from (5.8)} \\
|U^i|_j| = |A_k^i a_j^\ell \frac{\partial u_k}{\partial x_\ell}| = \mathcal{O}(\frac{d}{h} \varepsilon) \\
|U^i{}_{,j}| = \mathcal{O}(|U^i|_j| + |G_1 U|) = \mathcal{O}(\frac{d}{h} \varepsilon) = \mathcal{O}(\theta_0 \theta) \text{ from (5.3)} \\
|E_j^i| = |A_k^i a_j^\ell e_{k\ell}| = \mathcal{O}(\varepsilon) \text{ from (5.4)} \\
|J_1(U)| = |G_1 U + G_0 U'| = \mathcal{O}(\theta_0 \theta \frac{d}{h} \varepsilon) = \mathcal{O}(\varepsilon) \\
|R| = \mathcal{O}(\varepsilon + \frac{d^2}{h^2} \varepsilon^2) = \mathcal{O}(\varepsilon) \text{ from (5.12)} \\
|L| = \mathcal{O}(|E| + |R|) = \mathcal{O}(\varepsilon) \\
|M| = \mathcal{O}(|L|) = \mathcal{O}(\varepsilon) \\
|Z| = \mathcal{O}(|E|^2) = \mathcal{O}(\varepsilon^2) \\
|T| = \mathcal{O}(|E| + |Z|) = \mathcal{O}(\varepsilon) \\
\left.\begin{array}{l}
|P| = \mathcal{O}(|U'||T| + |T||J_1(U)|) = \mathcal{O}(\theta_0 \theta \varepsilon) \\
|Q| = \mathcal{O}(|T| + |P|) = \mathcal{O}(\varepsilon) \\
|S| = \mathcal{O}(|Q| + |M|) = \mathcal{O}(\varepsilon)
\end{array}\right\} \text{ from (5.12)}
\end{cases}
$$

From $|M| = \mathcal{O}(\varepsilon)$ and the appendix

$$(8.2)\begin{cases}
\|\zeta\dot{M}\| = \mathcal{O}(\frac{1}{d}\|M\|) + \sigma(d\|\zeta\ddot{M}\|) = \mathcal{O}(\varepsilon\sqrt{h}) + \sigma(dA_3) \\
|M_{,3}| = \mathcal{O}(\frac{\varepsilon}{h}) \\
\|\zeta M_{,3}\| = \mathcal{O}(\frac{\varepsilon d}{\sqrt{h}}) \\
\|\zeta M'\| = \mathcal{O}(\frac{\varepsilon d}{\sqrt{h}}) + \sigma(dA_3) \\
\|\zeta\ddot{M}\| = \mathcal{O}(\frac{1}{d^2}\|M\|) + \sigma(\|\zeta\ddot{M}\|) = \mathcal{O}(\frac{\sqrt{h}\varepsilon}{d}) + \sigma(A_3) \\
\|\zeta\dot{M}_{,3}\| = \mathcal{O}(\frac{\varepsilon}{\sqrt{h}}) \text{ since } |\zeta\dot{M}_{,3}| = \mathcal{O}(\frac{\varepsilon}{dh}) \\
\|\zeta\dot{M}'\| = \mathcal{O}(\frac{\varepsilon}{\sqrt{h}}) + \sigma(A_3) \\
\|\zeta\dot{M}^2\| = \mathcal{O}[\varepsilon(\|\zeta\ddot{M}\| + \frac{1}{d^2}\|M\|)] = \mathcal{O}[\theta^2(A_3 + \frac{\varepsilon\sqrt{h}}{d})] \\
\qquad = \mathcal{O}(\theta^2 A_3 + \frac{h^2\sqrt{h}\varepsilon}{d^3}) \\
\|\zeta M^2_{,3}\| = \mathcal{O}(\frac{\varepsilon\sqrt{h}}{d}) \text{ since } |M_{,3}| = \mathcal{O}(\frac{\varepsilon}{h}) \\
\|\zeta M'^2\| = \mathcal{O}(\theta^2 A_3 + \frac{\sqrt{h}\varepsilon}{d})
\end{cases}$$

If A and B are sets of functions, the notation A Coef(B) will be used to denote a linear combination of the functions from A with coefficients depending on the functions in B and on W. Then since $|E| = \mathcal{O}(\theta_0^2)$ from (8.1), Z^i_j is a quadratic function of the T^k_ℓ with coefficients depending on the T^k_ℓ and W for θ_0 small enough (see the discussion in section 5). Thus

$$(8.3)\quad \begin{cases} Z = T^2 \operatorname{Coef}(T) \\ Z' = TT' \operatorname{Coef}(T) \\ Z'' = (TT'' + T'^2) \operatorname{Coef}(T) \\ Z''' = (TT''' + T'T'' + T'^3) \operatorname{Coef}(T) \\ Z'''' = (TT'''' + T'T''' + T''^2 + T'^2T'' + T'^4) \operatorname{Coef}(T), \text{ etc.} \end{cases}$$

Next

$$(8.4)\quad \begin{cases} |Z'| = \mathcal{O}(|T||T'|) = \mathcal{O}(\theta^2|T'|) \text{ from } (8.3) \\ |U''| = \mathcal{O}(|M'|) \text{ from } (4.2) \\ |J_2(U)| = \mathcal{O}(|G_0U'' + G_1U' + G_2U|) \\ \qquad = \mathcal{O}(\theta_0\theta|M'| + \frac{\varepsilon}{d}) \\ R' = J_2(U) + U'U'' + U''J_1(U) + U'J_2(U) + J_1(U)J_2(U) \text{ from } (5.12) \\ |R'| = \mathcal{O}(\theta_0\theta|M'| + \frac{\varepsilon}{d}) \\ |T'| = \mathcal{O}(|M'| + |R'| + |Z'|) \text{ from } (5.12) \\ \qquad = \mathcal{O}(|M'| + \frac{\varepsilon}{d} + \theta_0^2|T'|) \\ |T'| = \mathcal{O}(|M'| + \frac{\varepsilon}{d}) \text{ for } \theta_0 \text{ small enough} \\ |Z'| = \mathcal{O}(\theta^2|M'| + \frac{h^2\varepsilon}{d^3}) \text{ from above} \\ |\dot{T}| = \mathcal{O}(|\dot{M}| + |R'| + |Z'|) \\ \qquad = \mathcal{O}(|\dot{M}| + \theta_0\theta|M'| + \frac{\varepsilon}{d}) \\ |\dot{P}| = \mathcal{O}[|U'||\dot{T}| + |U''||T| + |T||J_2(U)| + |\dot{T}||J_1(U)|] \text{ from } (5.12) \\ \qquad = \mathcal{O}(\theta_0\theta|\dot{M}| + \theta^2|M'| + \frac{h\varepsilon}{d^2}) \\ |P'| = \mathcal{O}(\theta_0\theta|M'| + \frac{h\varepsilon}{d^2}) \\ |S'| = \mathcal{O}(|R'| + |Z'| + |P'|) \text{ from } (5.12) \\ \qquad = \mathcal{O}(\theta_0\theta|M'| + \frac{\varepsilon}{d}) \\ |\dot{Q}| = \mathcal{O}(|\dot{M}| + |S'|) \text{ from } (5.12) \\ \qquad = \mathcal{O}(|\dot{M}| + \theta_0\theta|M'| + \frac{\varepsilon}{d}) \\ |Q'| = \mathcal{O}(|M'| + |S'|) \\ \qquad = \mathcal{O}(|M'| + \frac{\varepsilon}{d}) \end{cases}$$

Using (8.2) and (8.4)

$$
(8.5)\quad \begin{cases}
\|\zeta\dot{U}''\| &= \mathcal{O}(\|\zeta\dot{M}'\|) = \mathcal{O}(\frac{\varepsilon}{\sqrt{h}}) + \sigma(A_3) \\
\|\zeta\dot{T}\| &= \mathcal{O}(\varepsilon\sqrt{h}) + \sigma(dA_3) \\
\|\zeta\dot{P}\| &= \mathcal{O}(\frac{\varepsilon h\sqrt{h}}{d} + \theta_0\theta dA_3) \\
\|\zeta S'\| &= \mathcal{O}(\sqrt{h}\,\varepsilon + \theta_0\theta dA_3) \\
\|\zeta\dot{Q}\| &= \mathcal{O}(\varepsilon\sqrt{h} + dA_3) \\
\|\zeta Q'\| &= \mathcal{O}(\frac{\varepsilon d}{\sqrt{h}} + dA_3) \\
\|\zeta J_1(Q)\| &= \mathcal{O}[\theta_0\theta(\|\zeta Q'\| + \frac{1}{d}\|\zeta Q\|)] \\
&= \mathcal{O}(\theta_0\theta dA_3 + \frac{\varepsilon h\sqrt{h}}{d})
\end{cases}
$$

From (6.1)

$$
\begin{aligned}
\|\zeta V\| &= \mathcal{O}[\varepsilon\sqrt{h} + \|\zeta\dot{M}\| + \|\zeta S'\| + \|\zeta J_1(Q)\|] \\
&= \mathcal{O}(\varepsilon\sqrt{h} + d\theta_0\theta A_3) + \sigma(dA_3)\ .
\end{aligned}
$$

Then from (8.2), (8.5), and (4.1),

$$
(8.6)\quad \begin{cases}
A_2 &= \mathcal{O}(\|\zeta\dot{M}\| + \|\zeta V\| + \|\zeta S'\|) \\
&= \mathcal{O}(\varepsilon\sqrt{h} + dA_3) \\
B_2 &= \mathcal{O}(\|\zeta M'\| + \|\zeta V\| + \|\zeta S'\|) \\
&= \mathcal{O}(\frac{\varepsilon d}{\sqrt{h}} + dA_3)
\end{cases}
$$

Next

$$
(8.7)\left\{
\begin{aligned}
|Z''| &= \mathcal{O}(\theta^2|T''| + |T'|^2) \text{ from } (8.3)\\
&= \mathcal{O}(\theta^2|T''| + |M'|^2 + \frac{h^2\varepsilon}{d^4}) \text{ from } (8.4)\\
\|\zeta Z''\| &= \mathcal{O}(\theta^2\|\zeta T''\| + \theta^2 A_3 + \frac{\sqrt{h}\varepsilon}{d}) \text{ from } (8.2)\\
|U'''| &= \mathcal{O}(|M''|)\\
J_3(U) &= G_0U''' + G_1U'' + G_2U' + G_3U\\
|J_3(U)| &= \mathcal{O}[\theta_0\theta(|M''| + \frac{1}{d}|M'|) + \frac{\varepsilon}{d^2}]\\
R'' &= J_3(U) + U'U''' + U''^2 + U''J_2(U) + U'''J_1(U) + U'J_3(U)\\
&\quad + J_2^2(U) + J_1(U)J_3(U) \text{ from } (8.4)\\
|R''| &= \mathcal{O}[\theta_0\theta(|M''| + \frac{1}{d}|M'|) + \frac{\varepsilon}{d^2} + |M'|^2]\\
\|\zeta R''\| &= \mathcal{O}(\theta_0\theta B_3 + \frac{\varepsilon\sqrt{h}}{d}) \text{ from } (8.2)\\
\|\zeta T''\| &= \mathcal{O}[\|\zeta(M'' + R'' + Z'')\|]\\
&= \mathcal{O}(B_3 + \frac{\varepsilon\sqrt{h}}{d} + \theta_0^2\|\zeta T''\|)\\
\|\zeta T''\| &= \mathcal{O}(B_3 + \frac{\varepsilon\sqrt{h}}{d}) \text{ for } \theta_0 \text{ small enough}\\
\|\zeta Z''\| &= \mathcal{O}(\theta^2 B_3 + \frac{\sqrt{h}\varepsilon}{d})\\
P'' &= U'T'' + U''T' + U'''T + TJ_3(U) + T'J_2(U) + T''J_1(U)\\
|P''| &= \mathcal{O}(\theta_0\theta|T''| + |M'|^2 + \theta^2|M''| + \frac{\theta^2}{d}|M'| + \frac{h^2\varepsilon}{d^4})\\
\|\zeta P''\| &= \mathcal{O}(\theta_0\theta B_3 + \frac{\varepsilon\sqrt{h}}{d})\\
\|\zeta S''\| &= \mathcal{O}[\|\zeta(R'' + Z'' + P'')\|]\\
&= \mathcal{O}(\theta_0\theta B_3 + \frac{\varepsilon\sqrt{h}}{d})\\
\|\zeta J_2(Q)\| &= \mathcal{O}[\theta_0\theta(\|\zeta Q''\| + \frac{1}{d}\|\zeta Q'\| + \frac{1}{d^2}\|\zeta Q\|)]\\
&= \mathcal{O}(\theta_0\theta B_3 + \frac{\sqrt{h}\varepsilon}{d})\\
\|\zeta\dot{M}_{\alpha\beta,3}\| &= \mathcal{O}(\frac{1}{hd}\|M\| + \frac{d}{h}\|\zeta\ddot{M}\| + \frac{h}{d}\|\zeta M_{\alpha\beta,33}\|) \text{ from } (A.4)\\
&= \mathcal{O}(\frac{\varepsilon}{\sqrt{h}} + \frac{d}{h}A_3)\\
\|\zeta V'\| &= \mathcal{O}[\frac{\varepsilon}{\sqrt{h}} + \|\zeta\ddot{M}\| + \sum_{\alpha,\beta}\|\zeta\dot{M}_{\alpha\beta,3}\| + \|\zeta S''\| + \|\zeta J_2(Q)\|] \text{ from } (6.1)\\
&= \mathcal{O}(\frac{\varepsilon}{\sqrt{h}} + \frac{d}{h}A_3 + \theta_0\theta B_3)
\end{aligned}
\right.
$$

From (4.1)

$$B_3 = \mathcal{O}(\|\zeta\ddot{M}\| + \sum_{\alpha,\beta} \|\zeta\dot{M}_{\alpha\beta,3}\| + \|\zeta V'\| + \|\zeta S''\|)$$

$$= \mathcal{O}(\frac{\varepsilon}{\sqrt{h}} + \frac{d}{h} A_3 + \theta_0^2 B_3) \text{ from (8.7)}$$

Hence

$$(8.8) \qquad B_3 = \mathcal{O}(\frac{\varepsilon}{\sqrt{h}} + \frac{d}{h} A_3)$$

for θ_0 small enough.

Rewriting some results of (8.7) with the aid of (8.8) and deriving others

$$(8.9)\begin{cases}
\|\zeta M''\| = \mathcal{O}(B_3) = \mathcal{O}(\frac{\varepsilon}{\sqrt{h}} + \frac{d}{h} A_3) \\
\|\zeta R''\| = \mathcal{O}(\frac{\varepsilon\sqrt{h}}{d} + \theta_0^2 A_3) \\
\|\zeta Z''\| = \mathcal{O}(\theta_0 \theta A_3 + \frac{\varepsilon\sqrt{h}}{d}) \\
\|\zeta\ddot{T}\| = \|\zeta(\ddot{M} + R'' + Z'')\| \\
\qquad = \mathcal{O}(A_3 + \frac{\varepsilon\sqrt{h}}{d}) \\
\|\zeta T''\| = \mathcal{O}(\frac{d}{h} A_3 + \frac{\varepsilon}{\sqrt{h}}) \\
\ddot{P} = U'\ddot{T} + U''\dot{T} + U'''T + TJ_3(U) + \dot{T}J_2(U) + \ddot{T}J_1(U) \\
|\ddot{P}| = \mathcal{O}(\theta_0\theta|\ddot{T}| + \frac{h}{d}|M'|^2 + \frac{d}{h}|\dot{M}|^2 + \frac{\theta^2}{d}|M'| + \theta^2|M''| + \frac{h^2\varepsilon}{d^4}) \\
\|\zeta\ddot{P}\| = \mathcal{O}(\theta_0\theta A_3 + \frac{\varepsilon h\sqrt{h}}{d^2}) \\
\|\zeta S''\| = \mathcal{O}(\theta_0^2 A_3 + \frac{\varepsilon\sqrt{h}}{d}) \\
\|\zeta H_2(T)\| = \mathcal{O}[\theta_0\theta(\|\zeta\ddot{T}\| + \frac{1}{d}\|\zeta\dot{T}\| + \frac{1}{d}\|\zeta T\|)] \\
\qquad = \mathcal{O}(\theta_0\theta A_3 + \frac{\varepsilon h\sqrt{h}}{d^2}) \\
\|\zeta H_2(Q)\| = \mathcal{O}(\theta_0\theta A_3 + \frac{\varepsilon h\sqrt{h}}{d^2}) \\
\|\zeta H_2(P)\| = \mathcal{O}(\theta^2 A_3 + \frac{\varepsilon h^2\sqrt{h}}{d^3}) \\
\|\zeta J_2(Q)\| = \mathcal{O}(\theta_0^2 A_3 + \frac{\varepsilon\sqrt{h}}{d})
\end{cases}$$

By the divergence theorem

$$\iiint \zeta^2 G_0 \dddot{Q}\,\ddot{U} \Big| = \iiint [\zeta^2(G_1\ddot{U} + G_0\dddot{U})\ddot{Q} + \zeta\dot{\zeta}G_0\ddot{Q}\ddot{U}]\Big|$$

$$= \mathcal{O}[\theta_0^2\theta^2(\frac{1}{d^2}\|\zeta M'\|^2 + \|\zeta M''\|^2 + \|\dot{\zeta}M'\|^2)] + \sigma(\|\zeta\ddot{Q}\|^2)$$

$$= \mathcal{O}(\frac{h\varepsilon^2}{d^2} + \theta_0^4 A_3^2) + \sigma(A_3^2)$$

Hence

$$\iiint \zeta^2 H_3(Q)\ddot{U}\Big| = \iiint \zeta^2(G_0\dddot{Q} + G_1\ddot{Q} + G_2\dot{Q} + G_3 Q)\ddot{U}\Big|$$

$$= \mathcal{O}(\frac{h\varepsilon^2}{d^2} + \theta_0^4 A_3^2 + \frac{\theta_0^2\theta^2}{d^2}\|\zeta M'\|^2) + \sigma(A_3^2 + \|\zeta\ddot{Q}\|^2 + \frac{1}{d^2}\|\zeta\dot{Q}\|^2 + \frac{1}{d^4}\|\zeta Q\|^2)$$

$$= \mathcal{O}(\frac{h\varepsilon^2}{d^2} + \theta_0^4 A_3^2) + \sigma(A_3^2) .$$

Similarly

$$\iiint \zeta\dot{\zeta}G_0\dddot{Q}\,\dot{U}\Big| = \iiint [\zeta\dot{\zeta}(G_1\dot{U} + G_0\ddot{U})\ddot{Q} + (\dot{\zeta}^2 + \zeta\ddot{\zeta})G_0\ddot{Q}\,\dot{U}]\Big|$$

$$= \mathcal{O}[\theta_0^2\theta^2(\frac{1}{d^2}\|\dot{\zeta}\dot{U}\|^2 + \|\dot{\zeta}M'\|^2 + \frac{\varepsilon^2}{h})] + \sigma(\|\zeta\ddot{Q}\|^2)$$

$$= \mathcal{O}(\frac{h\varepsilon^2}{d^2}) + \sigma(A_3^2) ,$$

and

$$\left|\iiint \zeta\dot{\zeta}H_3(Q)\dot{U}\right| = \left|\iiint \zeta\dot{\zeta}(G_0\dddot{Q} + G_1\ddot{Q} + G_2\dot{Q} + G_3Q)\dot{U}\right|$$

$$= \mathcal{O}\left(\frac{h\varepsilon^2}{d^2} + \frac{\theta_0^2\theta^2}{d^2}\|\zeta\ddot{U}\|^2\right) + \sigma\left(A_3^2 + \|\zeta\ddot{Q}\|^2 + \frac{1}{d^2}\|\zeta\dot{Q}\|^2 + \frac{1}{d^4}\|\zeta Q\|^2\right)$$

$$= \mathcal{O}\left(\frac{h\varepsilon^2}{d^2}\right) + \sigma(A_3^2) \; .$$

Collecting terms, (7.1) becomes

$$\|\zeta\ddot{M}\| = \mathcal{O}\left(\frac{\varepsilon\sqrt{h}}{d} + \theta_0^2 A_3\right) + \sigma(A_3) \; .$$

From (6.1)

$$\left.\begin{array}{l}\|\zeta\dot{V}\| \\ \|\zeta V_{3,3}\|\end{array}\right\} = \mathcal{O}\left[\frac{\varepsilon\sqrt{h}}{d} + \|\zeta\ddot{M}\| + \|\zeta S''\| + \|\zeta J_2(Q)\|\right]$$

$$= \mathcal{O}\left(\frac{\varepsilon\sqrt{h}}{d} + \theta_0^2 A_3\right) + \sigma(A_3) \; .$$

From (4.1)

$$A_3 = \mathcal{O}(\|\zeta\ddot{M}\| + \|\zeta\dot{V}\| + \|\zeta V_{3,3}\| + \|\zeta S''\|)$$

$$= \mathcal{O}\left(\frac{\varepsilon\sqrt{h}}{d} + \theta_0^2 A_3\right) + \sigma(A_3) \; .$$

Therefore

$$(8.10)\qquad \begin{cases} A_3 = \mathcal{O}(\frac{\varepsilon\sqrt{h}}{d}) & \\ B_3 = \mathcal{O}(\frac{\varepsilon}{\sqrt{h}}) & \text{from (8.8)} \\ \left.\begin{array}{l} A_2 = \mathcal{O}(\varepsilon\sqrt{h}) \\ B_2 = \mathcal{O}(\frac{\varepsilon d}{\sqrt{h}}) \end{array}\right\} & \text{from (8.6)} \end{cases}$$

9. L_2 estimates for functions in a_4 and b_4.

Let $\zeta = \zeta_4$ in this section. From (5.15) $\zeta = \mathcal{O}(\zeta_3\zeta)$, $|\dot{\zeta}| = \mathcal{O}(\frac{1}{d}\zeta_3)$, and $|\ddot{\zeta}| = \mathcal{O}(\frac{1}{d^2}\zeta_3)$.

Let

$$(9.1)\qquad M_1 = \max \zeta|\dot{M}| \text{ and } M_2 = \max \zeta|M'| \ .$$

Then

$$
(9.2)\quad \left\{
\begin{array}{lll}
\left.\begin{array}{ll}
\zeta|\dot{T}| & = \mathcal{O}(M_1 + \theta_0\theta M_2 + \frac{\varepsilon}{d}) \\
\zeta|T'| & = \mathcal{O}(M_2 + \frac{\varepsilon}{d})
\end{array}\right\} & \text{from } (8.4) \\
\left.\begin{array}{ll}
\|\zeta_3\ddot{T}\| & = \mathcal{O}(\frac{\varepsilon\sqrt{h}}{d}) \\
\|\zeta_3 T''\| & = \mathcal{O}(\frac{\varepsilon}{\sqrt{h}})
\end{array}\right\} & \text{from } (8.9) \\
\left.\begin{array}{ll}
\|\zeta_3\dot{M}^2\| & = \mathcal{O}(\frac{\varepsilon h^2\sqrt{h}}{d^3}) \\
\|\zeta_3 M'^2\| & = \mathcal{O}(\frac{\varepsilon\sqrt{h}}{d})
\end{array}\right\} & \text{from } (8.2) \\
\left.\begin{array}{ll}
\|\zeta_3\dot{T}^2\| & = \mathcal{O}(\frac{\varepsilon h^2\sqrt{h}}{d^3}) \\
\|\zeta_3 T'^2\| & = \mathcal{O}(\frac{\varepsilon\sqrt{h}}{d})
\end{array}\right\} & \text{from } (8.4) \\
|J_1(U)| = \mathcal{O}(\varepsilon) \text{ from } (8.1) & & \\
|\zeta J_2(U)| = \mathcal{O}(\theta_0\theta M_2 + \frac{\varepsilon}{d}) \text{ from } (8.4) & & \\
\|\zeta_3 J_3(U)\| = \mathcal{O}(\frac{\sqrt{h}\,\varepsilon}{d}) \text{ from } (8.7)\ . & &
\end{array}\right.
$$

From (8.3)

$$
\|\zeta Z'''\| = \mathcal{O}(\theta^2\|\zeta T'''\| + \|\zeta T'T''\| + \|\zeta T'^3\|)\ .
$$

But

$$\|\zeta T'T''\| = \mathcal{O}(\|\zeta T'\zeta_3 T''\|) = \mathcal{O}[(M_2 + \tfrac{\varepsilon}{d})\|\zeta_3 T''\|]$$

$$= \mathcal{O}(\frac{\varepsilon}{\sqrt{h}}M_2 + \frac{\varepsilon h\sqrt{h}}{d^3}) \text{ from (9.2)}$$

and

$$\|\zeta T'^3\| = \mathcal{O}(\|\zeta T'\zeta_3 T'^2\|) = \mathcal{O}[(M_2 + \tfrac{\varepsilon}{d})\|\zeta_3 T'^2\|]$$

$$= \mathcal{O}(\frac{\varepsilon\sqrt{h}}{d}M_2 + \frac{\varepsilon h^2\sqrt{h}}{d^4}) \text{ from (9.2) .}$$

This type of reasoning is used repeatedly to obtain

$$\|\zeta Z'''\| = \mathcal{O}(\theta^2\|\zeta T'''\| + \frac{\varepsilon}{\sqrt{h}}M_2 + \frac{\varepsilon h\sqrt{h}}{d^3})$$

$$\|\zeta J_4(U)\| = \|\zeta(G_0U'''' + G_1U''' + G_2U'' + G_3U' + G_4U)\|$$

$$= \mathcal{O}[\theta_0\theta(B_4 + \frac{1}{d}B_3 + \frac{1}{d^2}B_2) + \frac{\varepsilon\sqrt{h}}{d^2}] \text{ from (8.1)}$$

$$= \mathcal{O}(\theta_0\theta B_4 + \frac{\varepsilon\sqrt{h}}{d^2}) \text{ from (8.10)}$$

$$R''' = J_4(U) + U'U'''' + U''U''' + U''J_3(U) + U'''J_2(U) + U''''J_1(U)$$
$$+ U'J_4(U) + J_2(U)J_3(U) + J_1(U)J_4(U) \text{ from (8.7)}$$

$$\|\zeta R'''\| = \mathcal{O}(\theta_0\theta B_4 + \frac{\varepsilon\sqrt{h}}{d^2} + \frac{\varepsilon}{\sqrt{h}}M_2)$$

$$\|\zeta T'''\| = \|\zeta(M''' + R''' + Z''')\|$$

$$= \mathcal{O}(B_4 + \frac{\varepsilon\sqrt{h}}{d^2} + \frac{\varepsilon}{\sqrt{h}}M_2 + \theta_0^2\|\zeta T'''\|)$$

$$\|\zeta T'''\| = \mathcal{O}(B_4 + \frac{\varepsilon\sqrt{h}}{d^2} + \frac{\varepsilon}{\sqrt{h}}M_2) \text{ for } \theta_0 \text{ small enough}$$

$$\|\zeta Z'''\| = \mathcal{O}(\theta^2 B_4 + \frac{\varepsilon}{\sqrt{h}}M_2 + \frac{\varepsilon h\sqrt{h}}{d^3})$$

$$P''' = U'T''' + U''T'' + U'''T' + U''''T + TJ_4(U) + T'J_3(U) + T''J_2(U)$$
$$+ T'''J_1(U)$$

$$\|\zeta P'''\| = \mathcal{O}(\theta_0\theta B_4 + \frac{\varepsilon h\sqrt{h}}{d^3} + \frac{\varepsilon}{\sqrt{h}}M_2)$$

$$\|\zeta S'''\| = \|\zeta(R''' + Z''' + P''')\|$$

$$= \mathcal{O}(\theta_0\theta B_4 + \frac{\varepsilon\sqrt{h}}{d^2} + \frac{\varepsilon}{\sqrt{h}}M_2)$$

$$\|J_3(Q)\| = \mathcal{O}[\theta_0\theta(\|\zeta Q'''\| + \frac{1}{d}\|\zeta Q''\| + \frac{1}{d^2}\|\zeta Q'\| + \frac{1}{d^3}\|\zeta Q\|)]$$

$$= \mathcal{O}[\theta_0\theta(B_4 + \frac{1}{d}B_3 + \frac{1}{d^2}B_2 + \frac{\varepsilon\sqrt{h}}{d^2})]$$

$$= \mathcal{O}(\theta_0\theta B_4 + \frac{\varepsilon\sqrt{h}}{d^2})$$

$$\|\zeta\ddot{M}_{\alpha\beta,3}\| = \mathcal{O}(\frac{1}{hd}\|\dot{M}\|_4 + \frac{d}{h}\|\zeta\dddot{M}\| + \frac{h}{d}\|\zeta\dot{M}_{\alpha\beta,33}\|) \text{ from (A.4)}$$

$$= \mathcal{O}(\frac{1}{hd}\|\zeta_3\dot{M}\| + \frac{d}{h}A_4)$$

$$= \mathcal{O}(\frac{d}{h}A_4 + \frac{\varepsilon}{d\sqrt{h}})$$

$$\|\zeta V''\| = \mathcal{O}(\frac{\varepsilon}{d\sqrt{h}} + \frac{d}{h}A_4 + \theta_0\theta B_4 + \frac{\varepsilon}{\sqrt{h}}M_2) \text{ from (6.1)}$$

From (4.1)

$$B_4 = \mathcal{O}(\|\zeta\dddot{M}\| + \sum_{\alpha,\beta}\|\zeta\ddot{M}_{\alpha\beta,3}\| + \|\zeta V''\| + \|\zeta S'''\|)$$

$$= \mathcal{O}(\frac{d}{h}A_4 + \frac{\varepsilon}{d\sqrt{h}} + \theta_0^2 B_4 + \frac{\varepsilon}{\sqrt{h}}M_2)$$

and for θ_0 small enough

(4.3) $$B_4 = \mathcal{O}(\frac{d}{h}A_4 + \frac{\varepsilon}{d\sqrt{h}} + \frac{\varepsilon}{\sqrt{h}}M_2) .$$

Rewriting some of the previous results and deriving others

$$\|\zeta M'''\| = \mathcal{O}(B_4) = \mathcal{O}\left(\frac{d}{h}A_4 + \frac{\varepsilon}{d\sqrt{h}} + \frac{\varepsilon}{\sqrt{h}}M_2\right)$$

$$\|\zeta J_4(U)\| = \mathcal{O}\left(\theta_0^2 A_4 + \frac{\varepsilon\sqrt{h}}{d^2} + \frac{\varepsilon\sqrt{h}}{d}M_2\right)$$

$$\|\zeta R'''\| = \mathcal{O}\left(\theta_0^2 A_4 + \frac{\varepsilon\sqrt{h}}{d^2} + \frac{\varepsilon}{\sqrt{h}}M_2\right)$$

$$\|\zeta Z'''\| = \mathcal{O}\left(\theta_0\theta A_4 + \frac{\varepsilon h\sqrt{h}}{d^3} + \frac{\varepsilon}{\sqrt{h}}M_2\right)$$

$$\|\zeta \dddot{T}\| = \|\zeta(\dddot{M} + R''' + Z''')\|$$

$$= \mathcal{O}\left(A_4 + \frac{\varepsilon\sqrt{h}}{d^2} + \frac{\varepsilon}{\sqrt{h}}M_2\right)$$

$$\|\zeta T''''\| = \mathcal{O}\left(\frac{d}{h}A_4 + \frac{\varepsilon}{d\sqrt{h}} + \frac{\varepsilon}{\sqrt{h}}M_2\right)$$

$$\dddot{P} = U'\dddot{T} + U''\ddot{T} + U'''\dot{T} + U''''T + TJ_4(U) + \dot{T}J_3(U) + \ddot{T}J_2(U) + \dddot{T}J_1(U)$$

$$\|\zeta\dddot{P}\| = \mathcal{O}\left(\theta_0\theta A_4 + \frac{\varepsilon\sqrt{h}\,h}{d^3} + \frac{\varepsilon\sqrt{h}}{d}M_2 + \frac{\varepsilon}{\sqrt{h}}M_1\right)$$

$$\|\zeta S'''\| = \mathcal{O}\left(\theta_0^2 A_4 + \frac{\varepsilon\sqrt{h}}{d^2} + \frac{\varepsilon}{\sqrt{h}}M_2\right)$$

$$\|\zeta H_3(T)\| = \mathcal{O}\left[\theta_0\theta\left(\|\zeta\dddot{T}\| + \frac{1}{d}\|\zeta\ddot{T}\| + \frac{1}{d^2}\|\zeta\dot{T}\| + \frac{1}{d^3}\|\zeta T\|\right)\right]$$

$$= \mathcal{O}\left(\theta_0\theta A_4 + \frac{\varepsilon h\sqrt{h}}{d^3} + \frac{\varepsilon\sqrt{h}}{d}M_2\right)$$

$$\|\zeta H_3(Q)\| = \mathcal{O}\left(\theta_0\theta A_4 + \frac{\varepsilon h\sqrt{h}}{d^3}\right)$$

$$\|\zeta H_3(P)\| = \mathcal{O}\left(\theta^2 A_4 + \frac{\varepsilon\sqrt{h}\,h^2}{d^4} + \frac{\varepsilon h\sqrt{h}}{d^2}M_2 + \frac{\varepsilon\sqrt{h}}{d}M_1\right)$$

$$\|\zeta J_3(Q)\| = \mathcal{O}\left(\theta_0^2 A_4 + \frac{\varepsilon\sqrt{h}}{d^2} + \frac{\varepsilon\sqrt{h}}{d}M_2\right)$$

Using the divergence theorem,

$$\left| \iiint \zeta^2 \dddot{U}\, G_0\, \ddddot{Q} \right| = \left| \iiint [\zeta^2(\ddddot{U}\, G_0 + \dddot{U}\, G_1) + \zeta\ \dot{\zeta}\ \dddot{U}\, G_0]\ \ddot{Q} \right|$$

$$= \mathcal{O}[\Theta_0^2\Theta^2(\|\zeta\ \ddddot{U}\|^2 + \|\dot{\zeta}\ \dddot{U}\|^2] + \sigma(\|\zeta\ \dddot{Q}\|^2) \ .$$

Since

$$\|\dot{\zeta}\ \dddot{U}\| = \mathcal{O}(\tfrac{1}{d}\ \|\zeta_3\ \dddot{U}\|) = \mathcal{O}(\tfrac{1}{d}\ B_3) = \mathcal{O}(\frac{\varepsilon}{d\sqrt{h}}) \ ,$$

the above becomes

$$\left| \iiint \zeta^2\ \dddot{U}\ G_0\ \overline{\overline{Q}} \right| = \mathcal{O}(\Theta_0^2\Theta^2 B_4^2 + \frac{h\varepsilon^2}{d^4}) + \sigma(A_4^2) \ ,$$

and

$$\left| \iiint \zeta^2 H_4(Q)\dddot{U} \right| = \mathcal{O}(\Theta_0^2\Theta^2 B_4^2 + \frac{h\varepsilon^2}{d^4} + \frac{\Theta_0^2\Theta^2}{d^2}\ \|\zeta\ \dddot{U}\|^2) + \sigma(A_4^2 + \|\zeta\dddot{Q}\|^2$$

$$+ \frac{1}{d^2}\ \|\zeta\ddot{Q}\|^2 + \frac{1}{d^4}\ \|\zeta\dot{Q}\|^2 + \frac{1}{d^6}\ \|\zeta Q\|^2)$$

$$= \mathcal{O}(\Theta_0^2\Theta^2 B_4^2 + \frac{h\varepsilon^2}{d^4}) + \sigma(A_4^2)$$

$$= \mathcal{O}(\frac{h\varepsilon^2}{d^4} + \Theta_0^4 A_4^2 + \frac{h\varepsilon^2}{d^2}\ M_2^2) + \sigma(A_4^2)$$

using (9.3).

Similarly

$$\left| \iiint \zeta\dot{\zeta}\, G_0 \ddddot{Q}\, \ddot{U} \right| = \left| \iiint [\zeta\dot{\zeta}(G_0\dddot{U} + G_1\ddot{U}) + (\zeta\ddot{\zeta} + \dot{\zeta}^2)G_0\ddot{U}]\, \dddot{Q} \right|$$

$$= \mathcal{O}[\Theta_0^2\Theta^2(\|\dot{\zeta}\, \dddot{U}\|^2 + \frac{1}{d^2}\|\dot{\zeta}\ddot{U}\|^2 + \frac{1}{d^4}\|\ddot{U}\|_4)] + \mathcal{O}(\|\zeta\, \dddot{Q}\|^2).$$

Since

$$\|\dot{\zeta}\, \ddot{U}\| = \mathcal{O}(\frac{1}{d}\, \|\zeta_2\ddot{U}\| = \mathcal{O}(\frac{1}{d}\, B_2) = \mathcal{O}(\frac{\varepsilon}{\sqrt{h}}) ,$$

and

$$\|\ddot{U}\|_4 = \mathcal{O}(\|\zeta_2\ddot{U}\|) = \mathcal{O}(B_2) = \mathcal{O}(\frac{d\varepsilon}{\sqrt{h}}) ,$$

the above becomes

$$\left| \iiint \zeta\, \dot{\zeta}\, G_0 \ddddot{Q}\, \ddot{U} \right| = \mathcal{O}(\frac{h\varepsilon^2}{d^4}) + \sigma(A_4^2) ,$$

and

$$\left| \iiint \zeta\, \dot{\zeta}\, H_4(Q)\ddot{U} \right| = \mathcal{O}(\frac{h\varepsilon^2}{d^4} + \frac{\Theta_0^2\Theta^2}{d^2}\, \|\dot{\zeta}\, \ddot{U}\|^2)$$

$$+ \sigma(A_4^2 + \|\zeta\, \dddot{Q}\|^2 + \frac{1}{d^2}\, \|\zeta\ddot{Q}\|^2 + \frac{1}{d^4}\|\zeta\dot{Q}\|^2 + \frac{1}{d^6}\|\zeta Q\|^2)$$

$$= \mathcal{O}(\frac{h\varepsilon^2}{d^4}) + \sigma(A_4^2 + \frac{1}{d^2}\, A_3^2 + \frac{1}{d^4}\, A_2^2)$$

$$= \mathcal{O}(\frac{h\varepsilon^2}{d^4}) + \sigma(A_4^2) .$$

Next

$$\|\dot{\zeta}\ \ddot{M}\| = \mathcal{O}(\tfrac{1}{d}\ \|\zeta_3\ddot{M}\|) = \mathcal{O}(\tfrac{1}{d}\ A_3) = \mathcal{O}(\frac{\sqrt{h}\ \varepsilon}{d^2})$$

$$\|\dot{M}\|_4 = \mathcal{O}(\|\zeta_2\dot{M}\|) = \mathcal{O}(A_2) = \mathcal{O}(\varepsilon\sqrt{h})$$

$$\|\dot{\zeta}\ U'''\| = \mathcal{O}(\tfrac{1}{d}\ \|\zeta_3 U'''\|) = \mathcal{O}(\tfrac{1}{d}\ B_3) = \mathcal{O}(\frac{\varepsilon}{d\sqrt{h}}) .$$

Collecting terms (7.1) becomes

$$\|\zeta\ \dddot{M}\| = (\frac{\sqrt{h}\ \varepsilon}{d^2} + \theta_0^2 A_4 + \frac{\varepsilon}{\sqrt{h}} M_2 + \frac{d\varepsilon}{h\sqrt{h}} M_1) + \sigma(A_4) .$$

From (6.1) and the above

$$\left.\begin{array}{l} \|\zeta\ddot{V}\| \\ \|\zeta V_{\alpha,33}\| \\ \|\zeta\dot{V}_{3,3}\| \end{array}\right\} = \mathcal{O}(\frac{\varepsilon\sqrt{h}}{d^2} + \theta_0^2 A_4 + \frac{\varepsilon}{\sqrt{h}} M_2 + \frac{d\varepsilon}{h\sqrt{h}} M_1) + \sigma(A_4).$$

From Theorem (4.1)

$$A_4 = \mathcal{O}(\|\zeta\ \dddot{M}\| + \|\zeta\ddot{V}\| + \|\zeta\dot{V}_{3,3}\| + \sum_{\alpha} \|\zeta V_{\alpha,33}\| + \|\zeta S'''\|)$$

$$= \mathcal{O}(\frac{\sqrt{h}\ \varepsilon}{d^2} + \theta_0^2 A_4 + \frac{\varepsilon}{\sqrt{h}} M_2 + \frac{d\varepsilon}{h\sqrt{h}} M_1) + \sigma(A_4) .$$

Thus

$$(9.4)\quad \begin{cases} A_4 = \mathcal{O}\left(\frac{\sqrt{h}\,\varepsilon}{d^2} + \frac{\varepsilon}{\sqrt{h}} M_2 + \frac{d\varepsilon}{h\sqrt{h}} M_1\right) \\ B_4 = \mathcal{O}\left(\frac{\varepsilon}{d\sqrt{h}} + \frac{\varepsilon d}{h\sqrt{h}} M_2 + \frac{d^2\varepsilon}{h^2\sqrt{h}} M_1\right) \end{cases}$$

from (9.3).

From (A.5)

$$\frac{\sqrt{h}}{d}\,|\zeta M'| = \mathcal{O}(\|\zeta M'''\| + \frac{1}{d^2}\|M'\|_4 + \frac{h^2}{d^2}\|\zeta M'''\|)$$

$$= \mathcal{O}(B_4 + \frac{1}{d^2} B_2) = \mathcal{O}\left(\frac{\varepsilon}{d\sqrt{h}} + \frac{\varepsilon d}{h\sqrt{h}} M_2 + \frac{d^2\varepsilon}{h^2\sqrt{h}} M_1\right)$$

$$= \mathcal{O}\left(\frac{\varepsilon}{d\sqrt{h}} + \frac{\theta_0\theta}{\sqrt{h}} M_2 + \frac{\theta_0^2}{\sqrt{h}} M_1\right)$$

so that

$$M_2 = \mathcal{O}(\frac{\varepsilon}{h} + \theta_0^2 M_2 + \theta_0^2 \frac{d}{h} M_1) \;;$$

$$(9.5)\qquad M_2 = \mathcal{O}(\frac{\varepsilon}{h} + \theta_0^2 \frac{d}{h} M_1) \qquad \text{for } \theta_0 \text{ small enough.}$$

Also from (A.5)

$$\frac{\sqrt{h}}{d}\,|\zeta \dot{M}| = \mathcal{O}(\|\zeta \dddot{M}\| + \frac{1}{d^2}\|\dot{M}\|_4 + \frac{h^2}{d^2}\|\zeta M'''\|)$$

$$= \mathcal{O}(A_4 + \frac{1}{d^2} A_2 + \frac{h^2}{d^2} B_4) = \mathcal{O}\left(\frac{\sqrt{h}\,\varepsilon}{d^2} + \frac{\varepsilon}{\sqrt{h}} M_2 + \frac{d\varepsilon}{h\sqrt{h}} M_1\right)$$

$$= \mathcal{O}\left(\frac{\sqrt{h}\,\varepsilon}{d^2} + \frac{\theta^2}{\sqrt{h}} M_2 + \frac{\theta_0\theta}{\sqrt{h}} M_1\right) .$$

$$M_1 = \mathcal{O}(\frac{\varepsilon}{d} + \theta_0\theta\, M_2 + \theta_0^2\, M_1) ,$$

$$= \mathcal{O}[\frac{\varepsilon}{d} + (\theta_0^4 + \theta_0^2)M_1]$$

using (9.5). Hence for θ_0 small enough

$$(9.6) \qquad \begin{cases} M_1 = \mathcal{O}(\frac{\varepsilon}{d}) \\ M_2 = \mathcal{O}(\frac{\varepsilon}{h}) \end{cases}$$

and from (9.4)

$$(9.3) \qquad \begin{cases} A_4 = \mathcal{O}(\frac{\sqrt{h}\,\varepsilon}{d^2}) \\ B_4 = \mathcal{O}(\frac{\varepsilon}{d\sqrt{h}}) . \end{cases}$$

10. Pointwise estimates for functions in a_k and b_k.

Continuing to let $\zeta = \zeta_4$, (A.5) gives

$$\frac{\sqrt{h}}{d}\ |\zeta S'| = \mathcal{O}(\|\zeta S'''\| + \frac{1}{d^2}\|S'\|_4) = \mathcal{O}(\frac{\sqrt{h}\,\varepsilon}{d^2}) ;$$

$$\max |\zeta S'| = \mathcal{O}(\frac{\varepsilon}{d}) .$$

Also

$$|\zeta J_1(Q)| = \mathcal{O}[\theta_0\theta(\max |\zeta Q'| + \frac{1}{d}\max |\zeta Q|)]$$
$$= \mathcal{O}(\frac{\varepsilon}{d} + \theta_0\theta \max |\zeta Q'|)$$

$$\max |\zeta J_1(Q)| = \mathcal{O}(\frac{\varepsilon}{d} + \theta_0\theta \max |\zeta Q'|) .$$

From (6.5) and (6.6)

$$|\zeta V| = \mathcal{O}[\max|\zeta\dot{M}| + \max|\zeta S'| + \max|\zeta J_1(Q)| + \max|F|]$$

$$= \mathcal{O}(M_1 + \frac{\varepsilon}{d} + \Theta_0\Theta \max|\zeta Q'|) \;;$$

$$\max|\zeta V| = \mathcal{O}(\frac{\varepsilon}{d} + \Theta_0\Theta \max|\zeta Q'|) \;.$$

From Theorem (4.1) for a in a_2,

$$|\zeta a| = \mathcal{O}(\max|\zeta\dot{M}| + \max|\zeta V| + \max|\zeta S'|)$$

$$= \mathcal{O}(\frac{\varepsilon}{d} + \Theta_0\Theta \max|\zeta Q'|)$$

and for b in b_2

$$|\zeta b| = \mathcal{O}(\frac{\varepsilon}{h} + \Theta_0\Theta \max|\zeta Q'|) \;.$$

Therefore

$$\max|\zeta Q'| = \mathcal{O}(\frac{\varepsilon}{h} + \Theta_0^2 \max|\zeta Q'|)$$

$$\max|\zeta Q'| = \mathcal{O}(\frac{\varepsilon}{h}) \quad \text{for } \Theta_0 \text{ small enough} \;,$$

so that

$$|\zeta a| = \mathcal{O}(\frac{\varepsilon}{d}) \quad \text{for a in } a_2,$$

and

$$|\zeta b| = \mathcal{O}(\frac{\varepsilon}{h}) \quad \text{for b in } b_2 \;.$$

Hence for $\sqrt{\theta_1^2 + \theta_2^2} \le d/4$ and $|\theta_3| \le h$,

$$(10.1) \qquad \begin{cases} |a| = \mathcal{O}(\frac{\varepsilon}{d}) & \text{for } a \text{ in } a_2 \\ |b| = \mathcal{O}(\frac{\varepsilon}{h}) & \text{for } b \text{ in } b_2 \,. \end{cases}$$

Since $\partial u_i/\partial x_j = \partial u_j/\partial x_i$ and $a^i_j = A^i_j = \delta^i_j$ at the origin, it follows that $U^i|_j = \partial u_i/\partial x_j$ and $U^i|_j = U^j|_i$ at the origin. Thus $U^i{}_{,j} = U^j{}_{,i} + J_1(U)$, $L_{ij} = \frac{1}{2}(U^i{}_{,j} + U^j{}_{,i}) = U^i{}_{,j} + J_1(U)$, and $|U'| = \mathcal{O}[\,|L|+|J_1(U)|\,] = \mathcal{O}(\varepsilon)$ at the origin.

By Taylor's theorem

$$U^\alpha{}_{,\beta} = U^\alpha{}_{,\beta}(0,0,0) + \theta_i U^\alpha{}_{,\beta i}(\bar\theta_1,\bar\theta_2,\bar\theta_3)$$

$$U^3{}_{,3} = U^3{}_{,3}(0,0,0) + \theta_i U^3{}_{,3i}(\theta_1^*,\theta_2^*,\theta_3^*) \,.$$

From (10.1) and the fact that $|U'| = \mathcal{O}(\varepsilon)$ at the origin,

$$\left.\begin{aligned} |U^\alpha{}_{,\beta}| &= \mathcal{O}(\varepsilon) \\ |U^3{}_{,3}| &= \mathcal{O}(\varepsilon) \end{aligned}\right\} \text{ for } \sqrt{\theta_1^2+\theta_2^2} \le \frac{d}{4} \text{ and } |\theta_3| \le h \,.$$
α

Since also $|M| = \mathcal{O}(\varepsilon)$, $|Q| = \mathcal{O}(\varepsilon)$, $|U'| = \mathcal{O}(\frac{d}{h}\,\varepsilon)$, it follows that

$$\left.\begin{aligned} |a| &= \mathcal{O}(\varepsilon) \quad \text{for } a \text{ in } a_1 \\ |b| &= \mathcal{O}(\tfrac{d}{h}\,\varepsilon) \quad \text{for } b \text{ in } b_1 \end{aligned}\right\} \text{ when } \sqrt{\theta_1^2+\theta_2^2} \le \frac{d}{4},\ |\theta_3| \le h \,.$$

Estimates can now be obtained for the L_2 norms of functions in a_5 and b_5 and pointwise estimates can then be obtained for the functions in a_3 and b_3, the procedure being similar to the work done in Sections 9 and 10. Continuing in this manner, pointwise estimates can be obtained for functions in a_k and b_k for k as large as one pleases.

Actually, the calculation of L_2 estimates for functions in a_k and b_k is easier for $k \geq 6$ since quantities such as M_1 and M_2 do not need to be introduced as in the cases $k = 4,5$. The pointwise estimates obtained are

$$|a| = \mathcal{O}\left(\frac{\varepsilon}{d^{k-1}}\right) \quad \text{for } a \text{ in } a_k$$

$$(10.2) \qquad |b| = \mathcal{O}\left(\frac{\varepsilon}{h\, d^{k-2}}\right) \quad \text{for } b \text{ in } b_k$$

for $k \geq 1$, $|\theta_3| \leq h$, and $\sqrt{\theta_1^2+\theta_2^2}$ small enough. The value of θ_0 and the constants in the order relations (10.2) depend of course on the largest value for k for which one wants (10.2) to be valid.

11. Special comparison of the shell theory with the classical theory, some improved estimates, and the importance of the low degree terms in the displacements.

The comparison with the classical theory made in this section is called special, because it is valid under more restrictive conditions than the comparison made in Section 3. The comparison is given by

$$(11.1)\begin{cases} \left|\dfrac{\partial^k(Q^\alpha{}_j|^j + F^\alpha)}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}\,\partial\theta_3^{k_3}}\right| = \mathcal{O}\begin{cases}(\frac{h}{d})^{n-k_3-1}\,\dfrac{\varepsilon}{d^{k+1}} & \text{if } n \text{ is odd}\\ (\frac{h}{d})^{n-k_3-2}\,\dfrac{\varepsilon}{d^{k+1}} & \text{if } n \text{ is even}\end{cases}\\ \text{if } n-k_3-1 \geq 0, \text{ and}\\ \left|\dfrac{\partial^k(Q^3{}_j|^j + F^3)}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}\,\partial\theta_3^{k_3}}\right| = \mathcal{O}\begin{cases}(\frac{h}{d})^{n-k_3}\,\dfrac{\varepsilon}{d^{k+1}} & \text{if } n \text{ is odd}\\ (\frac{h}{d})^{n-k_3-1}\,\dfrac{\varepsilon}{d^{k+1}} & \text{if } n \text{ is even}\end{cases}\\ \text{if } n-k_3 \geq 0\ . \end{cases}$$

$$(11.2)\begin{cases} \left|\dfrac{\partial^k(Q^\alpha{}_3 - \bar{Q}^\alpha)}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}}\right|_{\theta_3=\pm h} = \mathcal{O}\begin{cases}(\frac{h}{d})^{n}\,\dfrac{\varepsilon}{d^{k}} & \text{if } n \text{ is odd}\\ (\frac{h}{d})^{n-1}\,\dfrac{\varepsilon}{d^{k}} & \text{if } n \text{ is even}\end{cases}\\ \left|\dfrac{\partial^k(Q^3{}_3 - \bar{Q}^3)}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}}\right|_{\theta_3=\pm h} = \mathcal{O}\begin{cases}(\frac{h}{d})^{n+1}\,\dfrac{\varepsilon}{d^{k}} & \text{if } n \text{ is odd}\\ (\frac{h}{d})^{n}\,\dfrac{\varepsilon}{d^{k}} & \text{if } n \text{ is even}\end{cases} \end{cases}$$

all of which are valid for $|\theta_3| \leq h$ and for $\sqrt{\theta_1^2 + \theta_2^2}$ small enough.

The improved estimates of this section are

$$
(11.3)\quad \begin{cases}
\left| \dfrac{\partial^k Q^\alpha{}_3}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}} \right| = \mathcal{O}\left(\frac{h}{d}\,\frac{\varepsilon}{d^k}\right) \\[2ex]
\left| \dfrac{\partial^k Q^3{}_\alpha}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}} \right| = \mathcal{O}\left(\frac{h}{d}\,\frac{\varepsilon}{d^k}\right) \\[2ex]
\left| \dfrac{\partial^k (Q^3{}_3 - \bar{Q}^3)}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}} \right| = \mathcal{O}\left[\left(\frac{h}{d}\right)^2 \frac{\varepsilon}{d^k}\right] \\[2ex]
\left| \dfrac{\partial^k\, Q^3{}_{3,3}}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}} \right| = \mathcal{O}\left(\frac{h}{d}\,\frac{\varepsilon}{d^{k+1}}\right)
\end{cases}
$$

The importance of the low degree terms in the displacements and their derivatives is given by (11.4). Here $U^i_{[k]}$ is defined by $U^i = \sum_k U^i_{[k]} \left(\frac{\Theta_3}{h}\right)^k$.

$$
(11.4)\quad \begin{cases}
\left| \dfrac{\partial^\ell U^\alpha_{[k]}}{\partial\theta_1^{\ell_1}\,\partial\theta_2^{\ell_2}} \right| = \mathcal{O}\begin{cases} \left(\frac{h}{d}\right)^{k-1} \frac{\varepsilon}{d^{\ell-1}} & \text{for k odd} \\ \left(\frac{h}{d}\right)^{k} \frac{\varepsilon}{d^{\ell-1}} & \text{for k even} \end{cases} \\[4ex]
\left| \dfrac{\partial^\ell U^3_{[k]}}{\partial\theta_1^{\ell_1}\partial\theta_2^{\ell_2}} \right| = \mathcal{O}\begin{cases} \left(\frac{h}{d}\right)^{k} \frac{\varepsilon}{d^{\ell-1}} & \text{for k odd} \\ \left(\frac{h}{d}\right)^{k-1} \frac{\varepsilon}{d^{\ell-1}} & \text{for k even} \end{cases}
\end{cases}
$$

If a solution to the shell equations is substituted into the equilibrium equations (2.3) of the classical theory, then (11.1) gives estimates for the errors and those derivatives which do not have too many differentiations with respect to Θ_3. If the stresses, body forces, and their derivatives which appear in (11.1) are not of lower order than indicated by (10.2), (11.3), and (5.4), then (11.1) is significant if it gives lower order results than the aforementioned estimates. Since $Q^i{}_j|^j = Q^i_{j,j} + H_1(Q)$ from (6.4), it follows for example that when n = 3 the first estimate is significant for $k_3 = 0,1$ and the second estimate is significant for $k_3 = 0,1,2$. For the above cases the estimates are smaller by a factor of $(h/d)^2$ than the estimates using (10.2), (11.3), and (5.4). Furthermore, using this criterion, estimates (11.1) are not significant for any value of k_3 when n = 1,2.

Similarly if a solution to the shell equations is substituted into the boundary conditions (2.3) of the classical theory for $\Theta_3 = \pm h$, then (11.2) gives estimates for the errors and their derivatives. If the stresses, surface tractions, and their derivatives which appear in (11.2) are not of lower order than indicated by (11.3) and (5.4), then (11.2) is significant if it gives lower order results than the aforementioned estimates. Thus (11.2) is significant for n = 3, the estimates being smaller by a factor of $(h/d)^2$ than those obtained using (11.3) and (5.4). Again (11.2) is not significant, according to this criterion, if n = 1,2.

In the exceptional cases where the stresses, body forces, surface tractions, and their derivatives appearing in (11.1-2) are of lower order than indicated by (10.2), (11.3), and (5.4), the estimates (11.1-2) will still be significant for n large enough; however, barring these exceptional cases, (11.1-2) are significant for $n \geq 3$.

From (11.3) it is seen that the transverse shear stress $Q^{\alpha}{}_{3}$ and its derivatives tangent to the middle surface are of lower order than has been claimed for the other stresses and their derivatives tangent to the middle surface. Also the transverse normal stress $Q^{3}{}_{3}$ is approximately equal to the function which varies linearly between the prescribed transverse normal stresses at the faces, the same result being true for the derivatives of the transverse normal stress tangent to the middle surface. These results are analogous to results shown to be true for solutions to the classical theory when the surface tractions and body forces are zero [see p.3,ft.].

Finally (11.4) shows that the lower degree terms in the displacements and their derivatives will dominate the higher degree terms.

To prove (11.1) first observe that $Q^{i}{}_{j}|^{j} = Q^{i}_{j,j} + H_1(Q)$ from (6.4) and hence from (10.2)

$$\left|\frac{\partial^k Q^{i}{}_{j}|^{j}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}}\right| = \mathcal{O}\begin{cases} \dfrac{\varepsilon}{d^{k+1}} & \text{if } k_3 = 0, \text{ or} \\ & \text{if } i \neq 3 \text{ and } k_3 \text{ is even, or} \\ & \text{if } i = 3 \text{ and } k_3 \text{ is odd;} \\ \dfrac{\varepsilon}{h\ d^k} & \text{if } i \neq 3 \text{ and } k_3 \text{ is odd, or} \\ & \text{if } i = 3 \text{ and } k_3 \text{ is even.} \end{cases}$$

Using (5.4) and (5.10) in addition to the above

$$(11.5)\quad \left|\frac{\partial^k[(Q^i{}_j|^j+F^i)C]}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}}\right| = \begin{cases} \dfrac{\varepsilon}{d^{k+1}} & \text{if } k_3 = 0, \text{ or} \\ & \text{if } i \neq 3 \text{ and } k_3 \text{ is even, or} \\ & \text{if } i = 3 \text{ and } k_3 \text{ is odd;} \\ \dfrac{\varepsilon}{hd^k} & \text{if } i \neq 3 \text{ and } k_3 \text{ is odd, or} \\ & \text{if } i = 3 \text{ and } k_3 \text{ is even.} \end{cases}$$

From Theorem (3.1) and the mean value theorem, one obtains

$$\left|\frac{\partial^k[(Q^\alpha{}_j|^j+F^\alpha)C]}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}\,\partial\theta_3^{k_3}}\right| = \mathcal{O}\left\{h^{n-k_3-1}\max_{|\theta_3|\le h}\left|\frac{\partial^{k+n-k_3-1}[(Q^\alpha{}_j|^j+F^\alpha)C]}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}\,\partial\theta_3^{n-1}}\right|\right\}$$

if $n-k_3-1 > 0$, and

$$\left|\frac{\partial^k[(Q^3{}_j|^j+F^3)C]}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}\,\partial\theta_3^{k_3}}\right| = \mathcal{O}\left\{h^{n-k_3}\max_{|\theta_3|\le h}\left|\frac{\partial^{k+n-k_3}[(Q^3{}_j|^j+F^3)C]}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}\,\partial\theta_3^{n}}\right|\right\}$$

if $n-k_3 > 0$. These are also trivially true when $n-k_3-1 = 0$ and $n-k_3 = 0$ respectively.

Using (11.5) these become

$$(11.6)\quad \begin{cases} \left|\dfrac{\partial^k[(Q^\alpha{}_j|^j+F^\alpha)C]}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}}\right| = \mathcal{O}\begin{cases} \left(\frac{h}{d}\right)^{n-k_3-1}\dfrac{\varepsilon}{d^{k+1}} & \text{if } n \text{ is odd} \\ \left(\frac{h}{d}\right)^{n-k_3-2}\dfrac{\varepsilon}{d^{k+1}} & \text{if } n \text{ is even} \end{cases} \\ \text{if } n-k_3-1 \ge 0, \text{ and} \\ \left|\dfrac{\partial^k[(Q^3{}_j|^j+F^3)C]}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}\partial\theta_3^{k_3}}\right| = \mathcal{O}\begin{cases} \left(\frac{h}{d}\right)^{n-k_3}\dfrac{\varepsilon}{d^{k+1}} & \text{if } n \text{ is odd} \\ \left(\frac{h}{d}\right)^{n-k_3-1}\dfrac{\varepsilon}{d^{k+1}} & \text{if } n \text{ is even} \end{cases} \\ \text{if } n-k_3 \ge 0. \end{cases}$$

Now let n be odd. From (11.6)

$$|(Q^{\alpha}{}_{j}|^{j} + F^{\alpha})C| = \mathcal{O}[(\tfrac{h}{d})^{n-1} \tfrac{\varepsilon}{d}]$$

and

$$|Q^{\alpha}{}_{j}|^{j} + F^{\alpha}| = \mathcal{O}[(\tfrac{h}{d})^{n-1} \tfrac{\varepsilon}{d}]$$

since C > 1/2 (see Section 5). This is the first formula of (11.1) when k = 0 and n is odd.

Next for n odd

$$[(Q^{\alpha}{}_{j}|^{j} + F^{\alpha})C]_{,i} = (Q^{\alpha}{}_{j}|^{j} + F^{\alpha})_{,i}C + (Q^{\alpha}{}_{j}|^{j} + F^{\alpha})\, G_{i}.$$

Hence from (11.6) and the above

$$|(Q^{\alpha}{}_{j}|^{j}+F^{\alpha})_{,\beta}C| = \mathcal{O}[(\tfrac{h}{d})^{n-1} \tfrac{\varepsilon}{d^2}]\ ,\ |(Q^{\alpha}{}_{j}|^{j}+F^{\alpha})_{,3}C| = \mathcal{O}[(\tfrac{h}{d})^{n-2} \tfrac{\varepsilon}{d^2}],$$

and

$$|(Q^{\alpha}{}_{j}|^{j} + F^{\alpha})_{,\beta}| = \mathcal{O}[(\tfrac{h}{d})^{n-1} \tfrac{\varepsilon}{d^2}],\ |(Q^{\alpha}{}_{j}|^{j} + F^{\alpha})_{,3}| = \mathcal{O}[(\tfrac{h}{d})^{n-2} \tfrac{\varepsilon}{d^2}].$$

Proceeding in this manner the remainder of (11.1) can be proved.

From (2.7)

$$[(Q^{i}{}_{3} - \bar{Q}^{i})C]_{\theta_3=\pm h} = \frac{1}{2} \int_{-h}^{h} (Q^{i}{}_{j}|^{j} + F^{i})C(1 \pm \frac{\theta_3}{h})d\theta_3\ .$$

With this and (11.6), one has

$$
(11.7)\quad \begin{cases}
\left|\dfrac{\partial^k[Q^\alpha{}_3 - \bar{Q}^\alpha)C]}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}}\right|_{\theta_3=\pm h} = \mathcal{O}\begin{cases} (\frac{h}{d})^n \frac{\varepsilon}{d^k} & \text{if } n \text{ is odd} \\ (\frac{h}{d})^{n-1} \frac{\varepsilon}{d^k} & \text{if } n \text{ is even} \end{cases} \\[2ex]
\left|\dfrac{\partial^k[(Q^3{}_3 - \bar{Q}^3)C]}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}}\right|_{\theta_3=\pm h} = \mathcal{O}\begin{cases} (\frac{h}{d})^{n+1} \frac{\varepsilon}{d^k} & \text{if } n \text{ is odd} \\ (\frac{h}{d})^n \frac{\varepsilon}{d^k} & \text{if } n \text{ is even} \end{cases}
\end{cases}
$$

Then (11.2) follows from (11.7) in the same way that (11.1) followed from (11.6).

Next from (11.2) and (5.4)

$$
\left|\frac{\partial^k Q^\alpha{}_3}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}}\right|_{\theta_3=\pm h} = \mathcal{O}(\frac{h}{d}\frac{\varepsilon}{d^k}) \quad \text{for } n = 1,2,\ldots
$$

so that from Taylor's theorem

$$
\left|\frac{\partial^k Q^\alpha{}_3}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}}\right| = \mathcal{O}\left[\left|\frac{\partial^k Q^\alpha{}_3}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}}\right|_{\theta_3=h} + h\max_{|\theta_3|\leq h}\left|\frac{\partial^k Q^\alpha{}_{3,3}}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}}\right|\right]
$$

$$
= \mathcal{O}(\frac{h}{d}\frac{\varepsilon}{d^k})
$$

giving the first part of (11.3).

Using (10.2) and (5.12), it can be shown that $|\dot{P}^{(k)}| = \mathcal{O}(\frac{h}{d}\frac{\varepsilon}{d^k})$. Since $Q^i{}_j = T^i_j + P^i{}_j = Q_j{}^i + \dot{P}^i{}_j - P_j{}^i$, then $Q^i{}_j = Q^j{}_i + G_0 Q + P + G_0 P$. Thus

$$\frac{\partial^k Q^3{}_\alpha}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}} = \frac{\partial^k Q^\alpha{}_3}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}} + H_k(Q) + \dot{P}^{(k)} + H_k(P) ,$$

and the second part of (11.3) follows.

The third part of (11.3) follows from (11.2) and Taylor's theorem.

From $Q^3{}_3|^3 = Q^3{}_{3,3}$, (11.1), the second part of (11.3), and (5.4),

$$\left|\frac{\partial^k Q^3{}_{3,3}}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}}\right| = \mathcal{O}\left[\frac{h}{d}\,\frac{\varepsilon}{d^{k+1}} + \left|\frac{\partial^k (Q^3{}_\alpha|^\alpha + F^3)}{\partial\theta_1^{k_1}\,\partial\theta_2^{k_2}}\right|\right] = \mathcal{O}\left(\frac{h}{d}\,\frac{\varepsilon}{d^{k+1}}\right) .$$

This completes the proof of (11.3).

To obtain (11.4) one simply observes that

$$\frac{\partial^\ell U^i{}_{[k]}}{\partial\theta_1^{\ell_1}\,\partial\theta_2^{\ell_2}} = \frac{h^k}{k!}\,\left.\frac{\partial^{\ell+k}\,U^i}{\partial\theta_1^{\ell_1}\,\partial\theta_2^{\ell_2}\,\partial\theta_3^{k}}\right|_{\theta_3=0}$$

and uses (10.2) if $\ell+k > 0$, and $|U| = \mathcal{O}(\frac{d^2}{h}\,\varepsilon)$ if $\ell+k = 0$, $i = 3$.

If $k+\ell = 0$ and $i = \alpha$, (11.4) follows from

$$|U^\alpha| = \mathcal{O}[\,|\theta_i| \max_{|\theta_3|\le h} |U^\alpha{}_{,i}|\,] = \mathcal{O}(d\varepsilon) .$$

Appendix

Here are presented five trivial modifications of results from the appendix in [see p. 3, ft.], and an easily proved result for polynomials.

Let A be a finite set of functions of $\Theta_1, \Theta_2, \Theta_3$ defined in the region where $\zeta_k \neq 0$ and $|\Theta_3| \leq h$.
Let $\zeta = \zeta_k$ and $\bar{A} = \sup |A|$. Then

(A.1) $$\|\zeta\ddot{A}\| = \mathcal{O}\left(\frac{1}{d^2}\|A\|_k\right) + \sigma(\|\zeta\ \ddot{A}\|)$$

(A.2) $$\|\zeta\dot{A}\| = \mathcal{O}\left(\frac{1}{d}\|A\|_k\right) + \sigma(d\|\zeta\ \ddot{A}\|)$$

(A.3) $$\|\zeta\dot{A}^2\| = \mathcal{O}\left[\bar{A}\left(\|\zeta\ \ddot{A}\| + \frac{1}{d^2}\|A\|_k\right)\right]$$

(A.4) $$\|\zeta\dot{A}_{,3}\| = \mathcal{O}\left(\frac{1}{dh}\|A\|_k + \frac{d}{h}\|\zeta\ \ddot{A}\| + \frac{h}{d}\|\zeta A_{,33}\|\right)$$

(A.5) $$\frac{\sqrt{h}}{d}|\zeta A| = \mathcal{O}\left(\|\zeta\ \ddot{A}\| + \frac{1}{d^2}\|A\|_k + \frac{h^2}{d^2}\|\zeta A_{,33}\|\right)$$

(Sobolev's inequality).

If in addition the functions in A are all polynomials in Θ_3 whose degree depends only on n, then

(A.6) $$|A_{,3}| = \mathcal{O}\left(\frac{\bar{A}}{h}\right) \quad \text{for } |\Theta_3| \leq h.$$

To prove (A.6), let a be a polynomial in A whose degree is m (then m depends on n). Say

$$a = \sum_{i=0}^{m} b_i \left(\frac{\theta_3}{h}\right)^i .$$

Let a_i be the value of a for $\theta_3 = \frac{i}{m} h$ $(i = 0,1,\ldots,m)$. Then

$$a_i = \sum_{j=0}^{m} b_j \left(\frac{i}{m}\right)^j , \qquad i = 0,1,\ldots,m .$$

Treating the above as a system of linear equations with the b_j as unknowns, it is well known that the determinant of coefficients is not zero so that the b_j are linear combinations of the a_i. Since the coefficients of the above equations depend only on m (and hence only on n), each b_i is a linear combination of the a_j with coefficients depending only on n. Hence

$$|b_i| = \mathcal{O}\left(\max_{0 \le j \le m} |a_j|\right) = \mathcal{O}\left(\max_{|\theta_3| \le h} |a|\right) = \mathcal{O}(\bar{A}),$$

$$i = 0,1,\ldots,m.$$

But

$$|a_{,3}| = \left|\sum_{i=1}^{m} \frac{i}{h} b_i \left(\frac{\theta_3}{h}\right)^{i-1}\right| = \mathcal{O}\left(\frac{1}{h} \max_{0 \le i \le m} |b_i|\right) = \mathcal{O}\left(\frac{\bar{A}}{h}\right) ,$$

so that (A.6) is established.

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13. ABSTRACT

A non-linear theory for the equilibrium deformation of homogeneous isotropic shells is derived and compared with the classical three dimensional non-linear theory of elasticity.

To obtain the shell theory, let U^1 and U^2 be displacements tangent to the undeformed middle surface and let U^3 be a displacement normal to the middle surface. The displacement fields considered are restricted by requiring that U^1 and U^2 be nth degree polynomials in θ_3 and U^3 be an (n+1)st degree polynomial in θ_3 where θ_3 is the undeformed distance to the middle surface along a normal line. Restricting the displacements in this way, the potential energy of the shell becomes a functional of the 3n+4 coefficients of the idsplacement polynomials. Requiring that the potential energy be stationary with respect to variations of the 3n+4 coefficients, gives the equilibrium

(continued on page 85)

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13. Abstract - continued from page 83

equations and surface traction boundary conditions of the shell theory.

The shell theory is compared with the classical three dimensional theory by examining the errors which result when displacements satisfying the equilibrium equations and surface traction boundary conditions of the shell theory are substituted into those of the classical three dimensional theory. Let E_q, E_e and E_f denote any error resulting from an equilibrium equation, a surface traction boundary condition at the edge, and a surface traction boundary condition at a face, respectively, of the classical theory. It is shown that E_q, E_e and their derivatives have a stated number of zeros along each line normal to the middle surface, the number of zeros depending on n and the number of differentiations of the error with respect to θ_3. Furthermore, if the shell thickness and deformation are small enough, and if the body forces and surface tractions at the faces and their derivatives are small enough, then, at points not too near the edge, E_f and its derivatives are significantly small if n is large enough, and E_q and its derivatives are significantly small throughout the thickness if n is large enough and there are not too many differentiations of E_q with respect to θ_3. Also, under the previous restrictions, the low degree terms in the displacement polynomials and their derivatives are more significant than the high degree terms at points not too near the edge (at least this is always true if the difference in degrees of the two terms is greater than or equal two).